COS, Day 14

$f: X \to Y$ a continuous map of top. spaces.

$\rightsquigarrow \quad f_*: AbShv(X) \to AbShv(Y)$

is ~~right~~ left exact. Same for ringed spaces, $\mathcal{O}_X\text{-Mod} \to \mathcal{O}_Y\text{-Mod}$

As usual, for X, Y ~~reasonable~~ schemes, have

$$f_*: Qcoh(X) \to Qcoh(Y).$$

~~Both~~ All are left exact. Right derived functors are (abusively?) denoted $\mathbb{R}^i f_*$.

~~Fact When~~

<u>Fact</u> Whenever $Qcoh(X) \to AbShv(X)$ ~~preserves~~ takes injectives to acyclic objects for f_*, these agree.

E.g. X locally noetherian.

<u>Prop</u> $\mathbb{R}^i f_*(\mathcal{F})$ is the sheafification of the presheaf

$$V \longmapsto H^i(f^{-1}(V), \mathcal{F}|_{f^{-1}(V)}).$$

Pf. Define this sheaf to be $\mathcal{H}^i(X, \mathcal{F})$. Then $\{\mathcal{H}^i\}$ form a δ-functor, agreeing with f_* for $i=0$. Now if $I \in \mathcal{O}_X\text{-Mod}$ is injective, so is $I|_{f^{-1}(V)}$, so $\mathcal{H}^i(X, -)$ is effaceable. Thus they agree. $\square$

Cor $X \to \operatorname{Spec} R$ morphism of schemes, X locally noetherian. Then $\mathbb{R}^i f_* \mathcal{F} = H^i(X, \mathcal{F})$.

Cor If $f: X \to Y$ is morphism of schemes, X loc. noetherian, $\mathcal{F} \in \operatorname{QCoh}(X)$, then
$$\mathbb{R}^i f_* \mathcal{F} \in \operatorname{Qcoh}(Y).$$

How to Compute $\mathbb{R}^i f_* \mathcal{F}$

Given $X \xrightarrow{f} Y$ with X, Y separated, choose ~~affine~~ affine open cover $X = \bigcup \operatorname{Spec} R_i$.

Claim For $U \subseteq Y$ affine open, $f^{-1} U \cap \operatorname{Spec} R_i$ is affine open.

Pf. X, Y separated $\Longrightarrow$ $(1, f): X \longrightarrow X \times_{\mathbb{Z}} Y$ is closed immersion. (check!).

Now $f^{-1}(U) \cap \operatorname{Spec} R_i = (1, f)(X) \cap (\operatorname{Spec} R_i \times U)$ $\square$

Now form $f_* \check{C}_{sm}(\{\operatorname{Spec} R_i\}, \mathcal{F})$.

Claim $f_* \check{C}_{sm}(\{\operatorname{Spec} R_i\}, \mathcal{F}) \simeq \mathbb{R}^* f_* \mathcal{F}$.

Pf. By claim, restrict to Y affine. Now use that $\mathbb{R}^i f_* \mathcal{F} = H^i(X, \mathcal{F}) = H^i(f_* \check{C}_{sm}(\{\operatorname{Spec} R_i\}, \mathcal{F}))$

Rmk It's a Theorem of Grothendieck that:

Thm [Grothendieck] X a noetherian topological space of dimension n. Then for all $i > n$ and sheaves of abelian groups $\mathcal{F}$ on X,

$$H^i(X, \mathcal{F}) = 0.$$

We won't prove: see [H], Thm. III.2.7.

In general the relationship between

$$R^i f_* \mathcal{F} \text{ and, for some } y \in Y,$$

$$(R^i f_*)(\mathcal{F}|_{X_y}) \quad \text{is complicated!}$$

Lemma Suppose X is integral — for example an irreducible quasiprojective variety — and $X \xrightarrow{f} Y$ is dominant and Y is reduced (hence also integral). Then for any torsion-free quasicoherent sheaf $\mathcal{F}$ on X, $f_*\mathcal{F}$ is torsion-free.

Pf. Compute as $\ker\left(f_*\check{C}^0(\mathcal{U}, \mathcal{F}) \to f_*\check{C}^1(\mathcal{U}, \mathcal{F})\right)$, submodule of a torsion-free module. $\square$.

Example. E an elliptic curve, $L = \mathcal{O}_{E\times E}(E\times\{p\} - \Delta)$.

Consider $E\times E \xrightarrow{p_1} E$. What is $p_{1*}L$?

Over $(E \setminus \{p\}) \times E$, have $\overset{\text{divisor}}{(E \setminus \{p\}) \times \{p\}} - \Delta|_{(E\setminus\{p\})\times E}$,

disjoint union of two divisors.

Sections of $L|_{(E\setminus\{p\})\times E}$ restrict to each q

$\{q\} \times E$ as function with pole at

$\{q\} \times \{p\}$, zero at $\{q\} \times \{q\}$, regular

elsewhere. Now $H^0(E, \mathcal{O}(p-q)) = 0$ if $q \neq p$.

So, $f_* L|_{E \setminus \{p\}} = 0$. $(*)$

But, $L|_{\{p\}\times E} \cong \mathcal{O}_E$, so it has sections.

<u>Thus</u> $f_* L = 0$ (by $(*)$ plus lemma), while

$$f_*(L|_{\{p\}\times E}) = \mathcal{O}_p.$$

So if you hoped fibers of $f_* L$ computed by sections of $L|_{\{q\}\times E}$, they're not!

<u>Cor of Grothendieck's Thm</u>.

Suppose $f: X \to Y$ is a morphism with fibers of dim $\leq n$. Then $\mathbb{R}^n f_*$ is right-exact.

Pf. Use long exact coh. sequence plus Grothendieck. $\square$

Now if $\mathcal{F}$ is coh. sheaf on X, $y \in Y$ closed,

$X_y = X \times_Y \{y\} \subseteq X$ closed, get

$$\mathcal{F} \twoheadrightarrow \mathcal{F}|_{X_y}, \quad \text{hence}$$

$$R^n f_* \mathcal{F} \twoheadrightarrow R^n f_* \mathcal{F}|_{X_y}.$$

Commutative diagram

$$\begin{tikzcd}
X_y \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
\{y\} \arrow[r] & Y
\end{tikzcd}$$

tells us $R^n f_* \mathcal{F}|_{X_y} = H^n(X_y, \mathcal{F}|_{X_y})$.

<u>Conclusion</u> Analyze L on $E \times E$ again, where

$L = \mathcal{O}(E \times \{p\} - \Delta)$. Over $E \setminus \{p\}$,

compute $R^1 f_* L = 0$. <u>But</u>

$$R^1 f_* L \twoheadrightarrow R^1 f_* L|_{\{p\} \times E} = H^1(E, \mathcal{O}) = \mathcal{O}_p.$$

Math 595 Cohomology of Schemes, Day 15 Problems

Theorem 1 (Theorem on Formal Functions). *Let $f : X \to Y$ be a proper morphism of noetherian schemes, let $\mathcal{F}$ be a coherent sheaf on X, and let $y \in Y$. Then the natural map*

$$\widehat{R^i f_*(\mathcal{F})_y} \to \varprojlim H^i(X_n, \mathcal{F}_n),$$

where $X_n = X \times_Y \operatorname{Spec}(\mathcal{O}_y/\mathfrak{m}_y^n)$ and $\mathcal{F}_n = \mathcal{F}|_{X_n}$, is an isomorphism for all $i \geq 0$.

Corollary 2. *Let $f : X \to Y$ be a proper morphism of noetherian schemes, and let $r = \max\{\dim X_y \mid y \in Y\}$. Then $R^i f_* \mathcal{F} = 0$ for and $i > r$ and quasicoherent sheaves $\mathcal{F}$ on X.*

Recall from last time the invertible sheaf $L = \mathcal{O}(E \times \{p\} - \Delta)$ on $E \times E$, and its direct image $R^1 p_* L$ where $p : E \times E \to E$ is projection on the first factor.

Problem 3. Consider the neighborhood $(E \times E)_2$ of $(E \times E)_p = \{p\} \times E$. Show that $(E \times E)_2 \cong \operatorname{Spec}(k[\epsilon]) \times E$ where $k[\epsilon] := k[\epsilon]/(\epsilon^2)$ is the ring of dual numbers.

Then, choose such an isomorphism, inducing a projection $(E \times E)_2 \xrightarrow{q} E$, and form $q_* L_2$.

Problem 4. Show that $q_* L_2$ is a rank 2 locally free sheaf on E equipped with a short exact sequence

$$0 \to \mathcal{O}_E \cong q_*(\epsilon L_2) \to q_* L_2 \to q_*(L_2/\epsilon L_2) \cong \mathcal{O}_E \to 0.$$

Problem 5. Argue that the extension in the previous problem is non-split. [Your argument may have to be a little bit hand-wavy, I'm afraid.]

Problem 6. Use Serre duality to conclude that $R^1 p_* L_2$ has a 1-dimensional space of global sections. Hence, it is a skyscraper sheaf.

Problem 7. Use the Theorem on Formal Functions (and its corollary), the previous problem, and what we did last time to conclude that $R^1 p_* L \cong \mathcal{O}_p$ (the skyscraper sheaf at p).

More consequences of the Theorem on Formal Functions:

Corollary 8. *Let $f : X \to Y$ be a projective morphism of noetherian schemes, and assume that $f_* \mathcal{O}_X = \mathcal{O}_Y$. Then $f^{-1}(y)$ is connected, for every $y \in Y$.*

Theorem 9 (Zariski's Main Theorem). *Let $f : X \to Y$ be a birational projective morphism of noetherian integral schemes, and assume Y is normal. Then for every $y \in Y$, $f^{-1}(y)$ is connected.*

Theorem 10 (Stein Factorization). *Let $f : X \to Y$ be a projective morphism of noetherian schemes. Then f factors as $g \circ f'$ where $f' : X \to Y'$ is a projective morphism with connected fibers, and $g : Y' \to Y$ is a finite morphism.*

Problem 11. Suppose that $\{X_t\}$ is a flat family of closed subschemes of $\mathbb{P}^n_k$ parameterized by an irreducible nonsingular curve T of finite type over k. Suppose there is a nonempty open subset $U \subseteq T$ such that for all closed points $t \in U$, the fiber X_t is connected. Prove that X_t is connected for all $t \in T$.

Remark . Hartshorne states this for any integral curve T but I must admit I don't see how to prove it...do you?

COS, Day 16

<u>Semicontinuity Thm</u> Let X and Y be noetherian schemes. Let $f: X \to Y$ be a proper morphism, $\mathcal{F} \in \mathrm{coh}(X)$, $\mathcal{F}$ <u>flat over Y</u>. Then for each $i \geq 0$, the function

$$h^i(-, \mathcal{F}): Y \to \mathbb{Z}_{\geq 0}$$

$$h^i(y, \mathcal{F}) = \dim_{k(y)} H^i(X_y, \mathcal{F}_y)$$

is upper semicontinuous.

<u>Recall</u> $\mathcal{F}$ is flat over Y if for all $x \in X$, $\mathcal{F}_x$ is a flat $\mathcal{O}_{Y, f(x)}$-module.

h upper semicontinuous:

$h^{-1}(\mathbb{Z}_{\geq n})$ is closed for all $n \in \mathbb{Z}$.

"h^i can only jump up in limits."

<u>Ex</u> $L = \mathcal{O}(E \times \{p\} - \Delta)$ on $E \times E$ (Poincaré sheaf).

Then for $f = p_1: E \times E \to E$,

$$h^0(q, L) = h^1(q, L) = 0 \text{ for } q \neq p,$$
$$h^0(p, L) = h^1(p, L) = 1.$$

[Is L flat over E ?]

more precisely is L p_1-flat?

Theorem [Cohomology and Base Change]

Let $f: X \to Y$ be a proper morphism of noetherian schemes, $\mathcal{F} \in coh(X)$, $\mathcal{F}$ flat along f. Then

(i) if the natural map

$$\varphi^i(y): R^i f_* \mathcal{F} \otimes k(y) \longrightarrow H^i(X_y, \mathcal{F}_y)$$

is surjective, it is an isomorphism, and it is then an isomorphism for all y' in some neighborhood of y.

(ii) if $\varphi^i(y)$ is surjective, TFAE:

(a) $\varphi^{i-1}(y)$ is surjective

(b) $R^i f_* \mathcal{F}$ is locally free in a neighborhood of y.

The intuition and the proof are modelled on the situation where one has linear maps

$$k^{n_1} \xrightarrow{M_1} k^{n_2} \xrightarrow{M_2} k^{n_3}$$

where $M_1 = M_1(x_1, .., x_n)$, $M_2 = M_2(x_1 .., x_n)$ depend polynomially on parameters: so,

e.g. $C^\bullet: \left(\underset{0}{k[t]} \xrightarrow{(1,-1)} \underset{1}{k[t]^2} \xrightarrow{(t,t)} \underset{2}{k[t]} \right)$

In this example, $\ker(M_2) = \{(f,-f)\}$.

$M_2 \otimes k(0): k^2 \to k$ is the zero map.

So ~~[illegible]~~

$$H^2(C^\bullet \otimes k(0)) = k, \quad H^1(C^\bullet \otimes k(0)) = k^2/k\cdot(1,-1) \cong k.$$

Also ~~[illegible]~~ $H^2(C^\bullet) = k[t]/(t)$, get

$$\begin{array}{ccc} H^2(C^\bullet) \otimes k(0) & \longrightarrow & H^2(C^\bullet \otimes k(0)) \text{ isomorphism!} \\ \| & & \\ H^2(C^\bullet) & & \end{array}$$

$$H^1(C^\bullet) = \{(f,-f) \mid f \in k[t]\} / \mathrm{Im}(k[t]) = \{0\}.$$

Also $H^1(C^\bullet \otimes k(0)) = \ker(k^2 \xrightarrow{0} k) / \mathrm{Im}(k \xrightarrow{(1,-1)} k^2)$
$\cong k.$

So $H^1(C^\bullet \otimes k(0)) \longleftarrow H^1(C^\bullet) \otimes k(0)$ <u>not</u> surjective, as seen by $H^2(C^\bullet)$ not being locally free!