

COS, Day 13

Prop
~~Lemma~~

Let \mathcal{F} be a locally free $\mathcal{O}_{\mathbb{P}^1}$ -module of finite rank, for which:

(i) $\Gamma(\mathcal{F}(-1)) = \{0\}$,

(ii) $H^0(\mathcal{F}) \otimes_k \bigoplus_{\mathbb{P}^1} \mathcal{O}_{\mathbb{P}^1} \rightarrow \mathcal{F}$ is surjective.

Then $\mathcal{F} \cong \bigoplus_{\mathbb{P}^1}^{\text{rk}(\mathcal{F})} \mathcal{O}_{\mathbb{P}^1}$.

PF. By induction on $r = \text{rk}(\mathcal{F})$.

$r=1$ Choose any $s \in H^0(\mathcal{F}) \setminus \{0\}$

and consider

$$\mathcal{O} \xrightarrow{\cdot s} \mathcal{F}.$$

If $\cdot s$ is not surjective then s vanishes at some $p \in \mathbb{P}^1$, so s defines a nonzero section of $\mathcal{F}(-p)$, contradicting (i).

The map $\cdot s$ is nonzero since $s \neq 0$, and thus is injective since it is ~~generally~~ injective on every fiber. So $\cdot s$ is an isomorphism.

Inductive Step Arguing as in the $r=1$ case, there is a homomorphism

$$\mathcal{O} \xrightarrow{\cdot s} \mathcal{F}$$

that is injective and for which s is

nonzero in each fiber. By the proposition below, $\text{coker}(s)$ is locally free. We have an exact sequence $0 \rightarrow \mathcal{O} \rightarrow \mathcal{F} \rightarrow \text{coker}(s) \rightarrow 0$.

Assumption (ii) applies to $\text{coker}(s)$, so to apply the inductive hypothesis it suffices to show $H^0(\text{coker}(s)(-1)) = 0$. Using the LECs

we get
$$H^0(\mathcal{F}(-1)) \rightarrow H^0(\text{coker}(s)(-1)) \rightarrow$$

$\hookrightarrow H^1(\mathcal{O}(-1))$ exact, $H^0(\mathcal{F}(-1)) = 0$ by assumption, $H^1(\mathbb{P}^1, \mathcal{O}(-1)) = 0$ always, so

$H^0(\text{coker}(s)(-1)) = 0$. Now apply Ind. Hyp. to conclude $\text{coker}(s) \cong \mathcal{O}^{r-1}$, $r = \text{rk}(\mathcal{F})$. Let

(*) $0 \rightarrow \mathcal{O} \rightarrow \mathcal{F} \rightarrow \mathcal{O}^{r-1} \rightarrow 0$ exact. Now

$\text{Ext}_{\mathbb{P}^1}^1(\mathcal{O}^{r-1}, \mathcal{O}) \cong H^1(\mathbb{P}^1, \mathcal{O}^{r-1}) = 0$, so

(*) splits. This completes the inductive step, hence proof. \square

We used

Prop Suppose

$$0 \rightarrow \mathcal{G} \xrightarrow{\varphi} \mathcal{F} \xrightarrow{\psi} \mathcal{H} \rightarrow 0$$

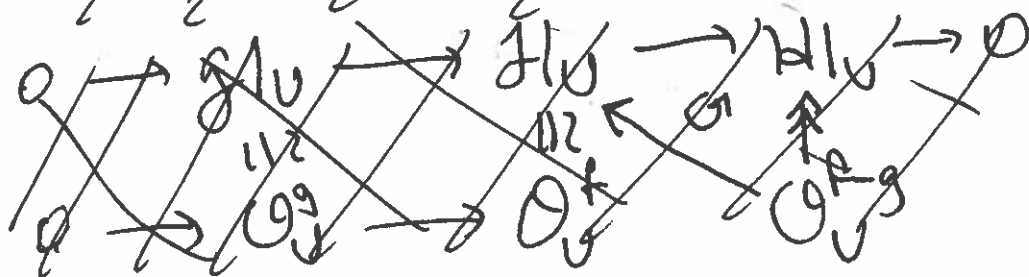
is an exact sequence of coherent sheaves on a scheme X with

- (a) \mathcal{G}, \mathcal{F} loc. free of ranks g, f .
- (b) for each $p \in X$,

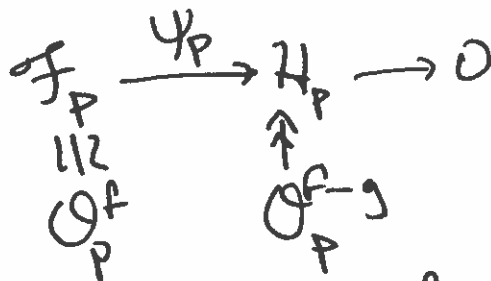
$g \otimes k(x) \xrightarrow{\psi_x} f \otimes k(x)$ is injective.

Then \mathcal{H} is locally free of rank $f-g$.

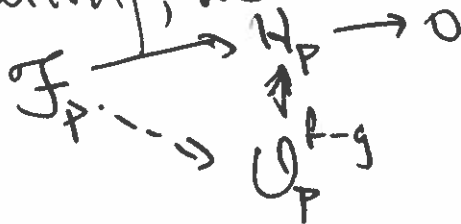
PF. It is a standard consequence of Nakayama's lemma that if $\dim_{k(x)} \mathcal{H}_x \otimes k(x) = m$ then \mathcal{H}_x is generated by m elements, in our case $f-g$ elements. ~~Fitting those to \mathcal{F}_x , there is an open nbhd $x \in U \subseteq X$ with~~



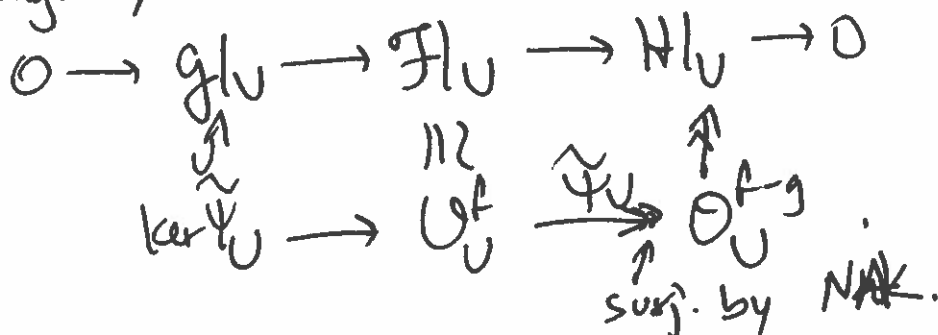
Now use



By projectivity, we can lift ψ_p to O_p^{f-g} , i.e.



Then there is some open nbhd $p \in U \subseteq X$ and a diagram,



Now applying $\otimes k(x)$ to above, we get

$$0 \rightarrow g \otimes k(p) \rightarrow \mathcal{F} \otimes k(p) \rightarrow \mathcal{H} \otimes k(p) \rightarrow 0$$

$$\uparrow \qquad \parallel \qquad \parallel$$

$$(\ker \tilde{\Psi}_U) \otimes k(p) \rightarrow k(p)^r \rightarrow (\ker \tilde{\Psi}_U)^{r-g} \rightarrow 0$$

with exact rows, so $(\ker \tilde{\Psi}_U) \otimes k(p) \rightarrow g \otimes k(p)$

is surjective, so again by NAK,

$\ker \tilde{\Psi}_U \rightarrow g$ is surjective. Both are loc. free of same rank, and again by NAK this must be an isomorphism. \square .

Back to \mathbb{P}^1 . Recall \mathcal{F} a loc. free \mathcal{O}_X -module of rank r .

let $m_1 = \max \{ m \mid H^0(\mathcal{F}(-m)) \neq 0 \}$.

Lemma $\mathcal{G}_1 := H^0(\mathcal{F}(-m_1)) \otimes \mathcal{O}(m_1) \rightarrow \mathcal{F}$
is injective w/ loc. free cokernel.

Pf. Equiv to consider $\mathcal{F}(-m_1)$, thus

$$H^0(\mathcal{F}(-m_1)) \otimes \mathcal{O} \rightarrow \mathcal{F}(-m_1)$$

with $H^0(\mathcal{F}(-m_1)(-1)) = 0$.

Let $g_1 = \mathcal{F}_1 \otimes \mathcal{O}(m_1)$

Then $H^0(g_1(-1)) \subseteq H^0(\mathcal{F}(-m_1)(-1)) = 0$.

So g_1 satisfies the hypotheses of today's first Prop. Hence

$H^0(\mathcal{F}(-m_1)) \otimes \mathcal{O} \rightarrow g_1$ is an isomorphism.

Also, $g_1 \otimes k(p) \rightarrow \mathcal{F}(-m_1) \otimes k(p)$ is injective for all $p \in \mathbb{P}^1$

if not, find $p \in \mathbb{P}^1$ and $s_p \in g_1 \otimes k(p)$ in kernel.

Then s_p lifts to $s \in H^0(g_1) = H^0(\mathcal{O}^{\text{rk}(g_1)})$,

and $s(p) = 0$ in $\mathcal{F} \otimes k(p)$. Thus

$s \in H^0(\mathcal{F}(-p))$, contradiction!

Now by 2nd Prop, $\mathcal{F}(-m_1)/g_1$ is loc free.

Iterating, we get a filtration

$$\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \dots \subseteq \mathcal{F}_l = \mathcal{F}$$

with each $\mathcal{F}_i/\mathcal{F}_{i-1}$ loc free,

$$\mathcal{F}_i/\mathcal{F}_{i-1} \cong \mathcal{O}(m_i)^{r_i}$$

Claim $m_1 > m_2 > \dots$

Pf. ~~Get~~ Get

$$0 \rightarrow H^0(\mathcal{F}_1(-m_1)) \xrightarrow{\sim} H^0(\mathcal{F}(-m_1)) \rightarrow H^0(\mathcal{F}/\mathcal{F}_1(-m_1))$$

$$\rightarrow H^1(\mathcal{F}_1(-m_1))$$

$$\parallel$$

$$H^1(\mathcal{O}^r)$$

$\Rightarrow H^0(\mathcal{F}/\mathcal{F}_1(-m_1)) = 0$, so $m_2 < m_1$. Iterate! \square

Finally,

Prop Given $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \dots \subseteq \mathcal{F}_\ell = \mathcal{F}$ on \mathbb{P}^1 ,

$$\mathcal{F}_i / \mathcal{F}_{i-1} \cong \mathcal{O}(m_i)^{r_i}, \quad m_1 > m_2 > \dots,$$

$$\text{have } \mathcal{F} \cong \bigoplus_{i=1}^{\ell} \mathcal{O}(m_i)^{r_i}.$$

Pf. Strong induction on rank. Clear for $r=1$.

$$\text{Using } 0 \rightarrow \mathcal{O}(m_1)^{r_1} \rightarrow \mathcal{F} \rightarrow \mathcal{F}/\mathcal{O}(m_1)^{r_1} \rightarrow 0,$$

Ind Hyp gives

$$\mathcal{F}/\mathcal{O}(m_1)^{r_1} \cong \bigoplus_{i=2}^{\ell} \mathcal{O}(m_i)^{r_i}.$$

Enough to show

$$\text{Ext}^1\left(\bigoplus_{i=2}^{\ell} \mathcal{O}(m_i)^{r_i}, \mathcal{O}(m_1)^{r_1}\right) = 0.$$

$$\text{But LHS} = \bigoplus_{i=2}^{\ell} H^1(\mathcal{O}(m_1)^{r_1} \otimes \mathcal{O}(-m_i)^{r_i})$$

$$= \bigoplus_{i=2}^{\ell} H^1(\mathcal{O}(m_1 - m_i)^{r_i}). \quad \text{Since } m_1 - m_i > 0,$$

this vanishes! □