COS, Day 13

Prop ~~Lemma~~ Let $\mathcal{F}$ be a locally free $\mathcal{O}_{\mathbb{P}^1}$-module of finite rank, for which:

(i) $\Gamma(\mathcal{F}(-1)) = \{0\}$.

(ii) $H^0(\mathcal{F}) \otimes_k \mathcal{O}_{\mathbb{P}^1} \to \mathcal{F}$ is surjective.

Then $\mathcal{F} \cong \mathcal{O}_{\mathbb{P}^1}^{rk(\mathcal{F})}$.

Pf. By induction on $r = rk(\mathcal{F})$.

<u>r=1</u> Choose any $s \in H^0(\mathcal{F}) \setminus \{0\}$ and consider

$$\mathcal{O} \xrightarrow{\cdot s} \mathcal{F}.$$

If $\cdot s$ is not surjective then s vanishes at some $p \in \mathbb{P}^1$, so s defines a nonzero section of $\mathcal{F}(-p)$, contradicting (i).

The map $\cdot s$ is nonzero since $s \neq 0$, and thus is injective since it is ~~generically~~ injective on every fiber. So $\cdot s$ is an isomorphism.

<u>Inductive Step</u> Arguing as in the $r=1$ case, there is a homomorphism

$$\mathcal{O} \xrightarrow{\cdot s} \mathcal{F}$$

that is injective and for which s is

nonzero in each fiber. By the proposition below, $\operatorname{coker}(s)$ is locally free. We have an exact sequence $0 \to \mathcal{O} \to \mathcal{F} \to \operatorname{coker}(s) \to 0$.
Assumption (ii) applies to $\operatorname{coker}(s)$, so to apply the inductive hypothesis it suffices to show $H^0(\operatorname{coker}(s)(-1)) = 0$. Using the LECS we get

$$H^0(\mathcal{F}(-1)) \to H^0(\operatorname{coker}(s)(-1)) \to H^1(\mathcal{O}(-1))$$

exact, $H^0(\mathcal{F}(-1)) = 0$ by assumption, $H^1(\mathbb{P}^1, \mathcal{O}(-1)) = 0$ always, so $H^0(\operatorname{coker}(s)(-1)) = 0$. Now apply Ind. Hyp. to conclude $\underset{\operatorname{coker}(s)}{\cancel{}} \cong \mathcal{O}^{r-1}$, $r = \operatorname{rk}(\mathcal{F})$. Get

$$(+) \quad 0 \to \mathcal{O} \to \mathcal{F} \to \mathcal{O}^{r-1} \to 0$$

exact. Now $\operatorname{Ext}^1_{\mathbb{P}^1}(\mathcal{O}^{r-1}, \mathcal{O}) \cong H^1(\mathbb{P}^1, \mathcal{O}^{r-1}) = 0$, so $(+)$ splits. This completes the inductive step, hence proof. $\square$

<u>We used</u>

<u>Prop</u> Suppose

$$0 \to \mathcal{G} \xrightarrow{\varphi} \mathcal{F} \xrightarrow{\psi} \mathcal{H} \to 0$$

is an exact sequence of coherent sheaves on a scheme X with

(a) $\mathcal{G}, \mathcal{F}$ loc. free of ranks g, f.

(b) for each $p \in X$,

$\mathcal{G}\otimes k(x) \xrightarrow{\varphi_x} \mathcal{F}_x\otimes k(x)$ is injective.

Then $\mathcal{H}$ is locally free of rank $f-g$.

Pf. It is a standard consequence of Nakayama's Lemma that if $\dim_{k(x)} \mathcal{H}_x\otimes k(x) = m$ then $\mathcal{H}_x$ is generated by m elements, in our case $f-g$ elements. ~~Lifting those to $\mathcal{F}_x$, there is an open nbhd $x\in U\subseteq X$ with~~

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{G}|_U \arrow[r] & \mathcal{F}|_U \arrow[r] & \mathcal{H}|_U \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{O}_U^g \arrow[r] \arrow[u, "\cong"] & \mathcal{O}_U^f \arrow[r] \arrow[u, "\cong"] & \mathcal{O}_U^{f-g} \arrow[u, two heads]
\end{tikzcd}$$

Now use

$$\begin{tikzcd}
\mathcal{F}_p \arrow[r, "\psi_p"] & \mathcal{H}_p \arrow[r] & 0 \\
\mathcal{O}_p^f \arrow[u, "\cong"] & \mathcal{O}_p^{f-g} \arrow[u, two heads]
\end{tikzcd}$$

By projectivity, we can lift ψ_p to $\mathcal{O}_p^{f-g}$, i.e.

$$\begin{tikzcd}
\mathcal{F}_p \arrow[r] \arrow[dr, dashed] & \mathcal{H}_p \arrow[r] & 0 \\
& \mathcal{O}_p^{f-g} \arrow[u, two heads]
\end{tikzcd}$$

Then there is some open nbhd $p\in U\subseteq X$ and a diagram,

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{G}|_U \arrow[r] & \mathcal{F}|_U \arrow[r] & \mathcal{H}|_U \arrow[r] & 0 \\
& \ker\tilde{\Psi}_U \arrow[r] \arrow[u, "\sim"] & \mathcal{O}_U^f \arrow[r, two heads, "\tilde{\Psi}_U"] \arrow[u, "\cong"] & \mathcal{O}_U^{f-g} \arrow[u, two heads]
\end{tikzcd}$$

surj. by NAK.

Now applying $\otimes k(x)$ to above, we get

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{G}\otimes k(p) \arrow[r] & \mathcal{F}\otimes k(p) \arrow[r] & \mathcal{H}\otimes k(p) \arrow[r] & 0 \\ & (\ker \tilde{\varphi}_U)\otimes k(p) \arrow[u] \arrow[r] & k(p)^f \arrow[u, "\cong"'] \arrow[r] & k(p)^{f-g} \arrow[u, "\cong"'] \arrow[r] & 0 \end{tikzcd}$$

with exact rows, so $(\ker \tilde{\varphi}_U)\otimes k(p) \to \mathcal{G}\otimes k(p)$ is surjective, so again by NAK, $\ker \tilde{\varphi}_U \to \mathcal{G}$ is surjective. Both are loc. free of same rank, and again by NAK this must be an isomorphism. $\square$.

Back to $\mathbb{P}^1$! <u>Recall</u> $\mathcal{F}$ a loc. free $\mathcal{O}_X$-module of rank r.

let $m_1 = \max\{m \mid H^0(\mathcal{F}(-m)) \neq 0\}$.

<u>Lemma</u> $\mathcal{F}_1 := H^0(\mathcal{F}(-m_1)) \otimes \mathcal{O}(m_1) \to \mathcal{F}$ is injective w/ loc. free cokernel.

Pf. Equiv to consider $\mathcal{F}(-m_1)$, thus $H^0(\mathcal{F}(-m_1))\otimes \mathcal{O} \to \mathcal{F}(-m_1)$ with $H^0(\mathcal{F}(-m_1)(-1)) = 0$.

Let $\mathcal{G}_1 =$ $\operatorname{Im}(H^0(\mathcal{F}(-m_1))\otimes \mathcal{O} \to \mathcal{F}(-m_1))$.

Then $H^0(\mathcal{G}_1(-1)) \subseteq H^0(\mathcal{F}(-m_1)(-1)) = 0$.

So $\mathcal{G}_1$ satisfies the hypotheses of today's first Prop. Hence

$H^0(\mathcal{F}(-m_1))\otimes\mathcal{O} \to \mathcal{G}_1$ is an isomorphism.

Also, $\mathcal{G}_1\otimes k(p) \to \mathcal{F}(-m_1)\otimes k(p)$ is injective for all $p\in\mathbb{P}^1$ —

if not, find $p\in\mathbb{P}^1$ and $s_p\in\mathcal{G}_1\otimes k(p)$ in kernel.

Then s_p lifts to $s\in H^0(\mathcal{G}_1)=H^0(\mathcal{O}^{rk(\mathcal{G}_1)})$,

and $s(p)=0$ in $\mathcal{F}\otimes k(p)$. Thus

$s\in H^0(\mathcal{F}(-p))$, contradiction!

Now by 2nd Prop, $\mathcal{F}(-m_1)/\mathcal{G}_1$ is loc free.

Iterating, we get a filtration

$$\mathcal{F}_1\subseteq\mathcal{F}_2\subseteq\dots\subseteq\mathcal{F}_l=\mathcal{F}$$

with each $\mathcal{F}_i/\mathcal{F}_{i-1}$ loc free,

$$\mathcal{F}_i/\mathcal{F}_{i-1}\cong\mathcal{O}(m_i)^{r_i}.$$

<u>Claim</u> $m_1>m_2>\dots$

Pf. Get

$$0\to H^0(\mathcal{F}_1(-m_1))\xrightarrow{\sim}H^0(\mathcal{F}(-m_1))\to H^0(\mathcal{F}/\mathcal{F}_1(-m_1))$$
$$\to H^1(\mathcal{F}_1(-m_1))$$
$$\cong$$
$$\cancel{H^1(\mathcal{O}^{r_1})}$$

$\Rightarrow H^0((\mathcal{F}/\mathcal{F}_1)(-m_1))=0$, so $m_2<m_1$. Iterate! $\square$

Finally,

Prop Given $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \dots \subseteq \mathcal{F}_\ell = \mathcal{F}$ on $\mathbb{P}^1$,

$\mathcal{F}_i/\mathcal{F}_{i-1} \cong \mathcal{O}(m_i)^{r_i}$, $m_1 > m_2 > \dots$,

have $\mathcal{F} \cong \bigoplus_{i=1} \mathcal{O}(m_i)^{r_i}$!

Pf. Strong induction on rank. Clear for $r=1$.

Using $0 \to \mathcal{O}(m_1)^{r_1} \to \mathcal{F} \to \mathcal{F}/\mathcal{O}(m_1)^{r_1} \to 0$,

Ind Hyp gives

$$\mathcal{F}/\mathcal{O}(m_1)^{r_1} \cong \bigoplus_{i=2}^{\ell} \mathcal{O}(m_i)^{r_i}.$$

Enough to show

$$\operatorname{Ext}^1\left(\bigoplus_{i=2}^{\ell} \mathcal{O}(m_i)^{r_i}, \mathcal{O}(m_1)^{r_1}\right) = 0.$$

But LHS $= \bigoplus_{i=2}^{\ell} H^1(\mathcal{O}(m_1)^{r_1} \otimes \mathcal{O}(-m_i)^{r_i})$

$= \bigoplus_{i=2}^{\ell} H^1(\mathcal{O}(m_1 - m_i))^{r_1 r_i}$! Since $m_1 - m_i > 0$,

this vanishes! $\square$