Math 595 Cohomology of Schemes, Day 12 Problems

Let $\mathcal{F}$ be a locally free sheaf of rank $r \geq 1$ on $\mathbb{P}^1_k$ (k a field).

Problem 1. Prove that there is some m_1 such that $\mathcal{F}(-m)$ has no global sections for $m > m_1$.

Let m_1 be maximal for which $\Gamma(\mathcal{F}(-m_1)) \neq 0$.

Problem 2. Show that the "evaluation map" $\mathcal{F}_1 := \Gamma(\mathcal{F}(-m_1)) \otimes_k \mathcal{O}(m_1) \to \mathcal{F}$ is injective, and that its cokernel is also a locally free sheaf on $\mathbb{P}^1_k$. [Careful, this does require some thought!]

Problem 3. Show that, if m_2 is maximal for which $\Gamma((\mathcal{F}/\mathcal{F}_1)(-m_2)) \neq 0$, then $m_1 > m_2$.

Problem 4. Prove by induction that $\mathcal{F}$ has a filtration

$$\mathcal{F}_0 = \{0\} \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_p = \mathcal{F},$$

with the property that each subquotient $\mathcal{F}_i/\mathcal{F}_{i-1}$ is isomorphic to some $\mathcal{O}(m_i)^{r_i}$ with $m_1 > m_2 > \cdots > m_p$, $r_i \geq 1$.

Problem 5. Prove by induction that $\mathcal{F} \cong \oplus \mathcal{F}_i/\mathcal{F}_{i-1}$. Hint: you will have to use the vanishing of some Ext^1 groups!

Remark . Note that the filtration

$$\mathcal{F}_0 = \{0\} \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_p = \mathcal{F}$$

was canonical, but that the splitting $\mathcal{F} \cong \oplus \mathcal{F}_i/\mathcal{F}_{i-1}$ is not unless $r = 1$.

Problem 6. Let $p \in \mathbb{P}^2_k$ be any point (k a field) and let $I_p \subset \mathcal{O}_{\mathbb{P}^2}$ denote its ideal sheaf. Prove

(1) that there is a nonsplit extension

$$0 \to \mathcal{O}_{\mathbb{P}^2_k} \to \mathcal{E} \to I_p \to 0,$$

(2) that such an extension is unique up to isomorphism as coherent sheaves (not as 1-extensions!), and

(3) that $\mathcal{E}$ is a locally free sheaf (necessarily of rank 2).

Prove further that $\mathcal{E}$ is *semistable* of slope 0, in the sense that if $0 \subsetneq \mathcal{F} \subsetneq \mathcal{E}$ is any nonzero proper coherent subsheaf, then $\deg(\mathcal{F}) \leq 0$.