Last Time we discussed Serre Duality for projective space.

General $F, G$ coherent sheaves on $\mathbb{P}^n$. Then

\[ \Ext^i_{\mathbb{P}^n}(F, G) = \Ext^{n-i}(g, F \otimes \Omega^m_{\mathbb{P}^n}) \text{ for } i \geq 0. \]

We prove it for $F = \mathcal{O}_{\mathbb{P}^n}$, but generalizes this way.

What about for other schemes?

**Ext Sheaves**

If $F \in \text{Qcoh}(X)$. Get a functor

\[ \text{Hom}(F, -) = \text{Hom}(F, -) : \text{Qcoh}(X) \rightarrow \text{Qcoh}(X), \]

in various "obvious" definitions.

**Def.** Sheaf Ext or Ext sheaves are

\[ \Ext^i(F, -) = \Ext^i(F, -) = R^i \text{Hom}(F, -). \]

**Prop.** For any open $U \subseteq X$,

\[ \Ext^i_X(F, G)|_U = \Ext^i_U(F|_U, G|_U). \]

**Pf.** Both are $S$-functors on $\mathcal{O}_U$-Mod, agree on $i = 0$, die for $F$ injective, done!

**Cor.** For $U = \text{Spec } R$ affine open, $F = \tilde{F}$, $G = \tilde{G}$,

\[ \Ext^i_U(F, G)(U) = \Ext^i_R(F, G). \]
Prop. If \( \cdots \to L_1 \to L_0 \to I \to 0 \) is an exact sequence in \( \mathcal{A} \)-sheaves with each \( L_i \) locally free of finite rank, then
\[
\text{Ext}^i(I, J) \cong H^i(\text{Hom}(L_0, J)).
\]
Pr. As usual. \( \square \)

Example. Suppose \( X \subseteq \mathbb{P}^n \) is a closed subscheme, \( I_X = (f_1, \ldots, f_r) \) a regular sequence for some admissible open \( U \subseteq \mathbb{P}^n \). Koszul complex
\[
K(f_1, \ldots, f_r)_{\bullet} \to \mathcal{O}_U
\]
gives \( \text{Ext}^i_U(\mathcal{O}_X, \mathcal{O}_{\mathbb{P}^n}) \cong H^i(K(f_1, \ldots, f_r)^\ast) \).

Because sequence is regular, this only has cohomology at \( i = r \). Get that
\[
\text{Ext}^i_U(\mathcal{O}_X, \mathcal{O}_{\mathbb{P}^n}) = 0 \text{ if } i \neq r,
\]
and
\[
H^r(K(f_1, \ldots, f_r)^\ast) \cong \bigwedge^r_{\mathcal{O}_X}(I_X/I_X^2)^\ast.
\]

Def. Write \( \omega_X = \text{Ext}^i(\mathcal{O}_X, \mathcal{O}_{\mathbb{P}^n}) \) for \( X \subseteq \mathbb{P}^n \).

Cohen-Macaulay --- do you know what that means?
Suppose $X$ is a projective, Cohen–Macaulay scheme. Then for coherent $F \in \text{Coh}(X)$, there are isomorphisms, functorial in $F$,

$$\text{Ext}^i_X(F, \omega_X) \cong H^{2d-i}(X, F)^*,$$

for all $i$, where $D = \dim X$.

The proof is to reduce to $\mathbb{P}^n$ case. See [H] § III.7.

Again, can extend to some coherent sheaves. I'm not actually sure offhand of the correct generality — I encourage you to try to look it up, since it will give you some experience with searching the literature. Try the Stacks Project.

Here's a statement that's OK for sure. Suppose $F, G \in \text{Coh}(X)$ and $G$ is perfect,

$$0 \to L_n \to L_{n-1} \to \ldots \to L_0 \to G \to 0$$

exact with each $L_i$ locally free of finite rank. Then

$$\text{Ext}^i_X(F, G \otimes \omega_X) \cong \text{Ext}^{D-i}_X(G, F)^*$$

for all $i$, where $D = \dim X$ (and again $X$ is projective and CM).
\[ \text{Ext}^1 \mathcal{F}, \mathcal{F}' \in \text{Qcoh}(X). \] An extension or better 1-extension of \( \mathcal{F}' \) by \( \mathcal{F}'' \) is a short exact sequence in \( \text{Qcoh}(X) \),
\[ 0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0. \] (t)

An isomorphism of 1-extensions is an isomorphism of short exact sequences that is the identity map on \( \mathcal{F}' \) and \( \mathcal{F}'' \).

Given an extension \((t)\), get boundary map
\[ \text{Hom}(\mathcal{F}'', \mathcal{F}') \to \text{Ext}^1(\mathcal{F}'', \mathcal{F}''). \]
\[ \text{sl}(\text{id}(\mathcal{F}'')) \text{ is the extension class of } (t). \]

**Prop** The above construction yields a bijection between the set of isomorphism classes of 1-extensions and \( \text{Ext}^1(\mathcal{F}'', \mathcal{F}') \).

**Pf** vague sketch. Choose an injective resolution
\[ \mathcal{F}' \to I^0 \to I^1 \to I^2 \ldots \]

Given a class in \( \text{Ext}^1(\mathcal{F}'', \mathcal{F}') \), choose representative \( \mathcal{F}' \xrightarrow{\eta} I^1 \) with \( \partial \eta = 0 \). So \( \eta(\mathcal{F}'') \subseteq \text{Im}(\partial) \)

Form pull back
\[ 0 \to \eta \to I^0 \xrightarrow{\eta} \mathcal{F}' \to \mathcal{F}'' \to 0 \]
\[ 0 \to \mathcal{F}' \to I^0 \xrightarrow{\eta} \text{Im}(\partial^0) \to 0 \]

to get 1-extension. Check these two constructions are mutually inverse.
Sometimes this information is overly refined; you might prefer to calculate
\[ \text{Aut}(\mathfrak{f}) \setminus \text{Ext}^1(\mathfrak{g}''', \mathfrak{g})/\text{Aut}(\mathfrak{g}''') \],

to get set of iso classes of exact sequences
\[ 0 \to \mathfrak{g}' \to \mathfrak{g} \to \mathfrak{g}'' \to 0 \]
with \( \mathfrak{g}' \cong \mathfrak{g}'', \mathfrak{g}'' \cong \mathfrak{g}' \).

BTW, above Prop was really an abelian category statement (if it has enough injectives).

\[ \text{Ex} X = \mathbb{A}^2_k \]
\[ \mathfrak{f}''' = (\mathfrak{g}_p, \mathfrak{g} = (0, 0), \]
\[ \mathfrak{g}'' = I_p, \]

Use
\[ 0 \to k[[xy]] \to k[[xy]] \to k[[xy]] \to k[[xy]]/(x, y) \to 0 \]

to get
\[ \text{Ext}^1_{k[[xy]]}(\otimes k[[xy]]/(x, y), (x, y)) \]
\[ = H^1(\text{Hom}(k[[xy]], (x, y)) \to \text{Hom}(k[[xy]]^2, (x, y)) \]
\[ \to \text{Hom}(k[[xy]], (x, y)) \), i.e of \]
\[ (x, y) \to (x, y)^2 \to (x, y) \]

What is \( H^1 \)? You compute!