COS, Day 11

Last Time We discussed Serre Duality for projective space.

General $\mathcal{F}, \mathcal{G}$ coherent sheaves on $\mathbb{P}^n$. Then
$$\mathrm{Ext}^i_{\mathbb{P}^n}(\mathcal{F}, \mathcal{G}) \cong \mathrm{Ext}^{n-i}(\mathcal{G}, \mathcal{F} \otimes \Omega^n_{\mathbb{P}^n})^* \quad \forall i.$$
We proved it for $\mathcal{F} = \mathcal{O}_{\mathbb{P}^n}$, but generalizes this way.

What about for other schemes?

Ext Sheaves

$\mathcal{F} \in \mathrm{Qcoh}(X)$. Get a functor
$$\mathcal{H}om(\mathcal{F}, -) = \underline{\mathrm{Hom}}(\mathcal{F}, -) : \mathrm{Qcoh}(X) \to \mathrm{Qcoh}(X),$$
in various "obvious" definitions.

Def Sheaf Ext or Ext sheaves are
$$\mathcal{E}xt^i(\mathcal{F}, -) = \underline{\mathrm{Ext}}^i(\mathcal{F}, -) = R^i \underline{\mathrm{Hom}}(\mathcal{F}, -).$$

Prop For any open $U \subseteq X$,
$$\mathcal{E}xt^i_X(\mathcal{F}, \mathcal{G})|_U \cong \mathcal{E}xt^i_U(\mathcal{F}|_U, \mathcal{G}|_U).$$

Pf. Both are δ-functors on $\mathcal{O}_U$-Mod, agree on $i=0$, die for $\mathcal{G}$ injective, done!

Cor For $U = \mathrm{Spec}\, R$ affine open, $\mathcal{F} = \tilde{F}$, $\mathcal{G} = \tilde{G}$,
$$\underline{\mathrm{Ext}}^i_U(\mathcal{F}, \mathcal{G})(U) = \mathrm{Ext}^i_R(F, G).$$

Prop If $\cdots \to L_1 \to L_0 \to \mathcal{F} \to 0$
is an exact sequence in $\mathcal{O}_X$-Mod with each L_i locally free of finite rank, then

$$\mathcal{E}xt^i(\mathcal{F}, \mathcal{G}) \cong \underline{H}^i(\mathcal{H}om(L_\bullet, \mathcal{G})).$$

Pf. As usual. $\square$.

Example Suppose $X \subseteq \mathbb{P}^n$ is a closed subscheme, $I_X|_U = (f_1, .., f_r)$ a regular sequence for some affine open $U \subseteq \mathbb{P}^n$. Koszul complex

$$K(f_1, .., f_r)_\bullet \longrightarrow \mathcal{O}_X$$

gives
$$\mathcal{E}xt^i_U(\mathcal{O}_X, \mathcal{O}_{\mathbb{P}^n}) \cong H^i(K(f_1 .. f_r)^\vee).$$

Because sequence is regular, this only has cohomology at $i = r$. Get that

$$\mathcal{E}xt^i_U(\mathcal{O}_X, \mathcal{O}_{\mathbb{P}^n}) = 0 \quad \text{if } i \neq r,$$
$$= H^r(K(f_1 .. f_r)^\vee) \text{ otherwise},$$

and
$$H^r(K(f_1 ... f_r)^\vee) \cong \Lambda^r_{\mathcal{O}_X}(I_X/I_X^2)^\vee.$$

Def Write $\omega_X = \mathcal{E}xt^i(\mathcal{O}_X, \mathcal{O}_{\mathbb{P}^n})$ for $X \subseteq \mathbb{P}^n$ Cohen-Macaulay – do you know what that means?

Thm Suppose X is a projective, Cohen–Macaulay scheme. Then for coherent $\mathcal{F} \in \mathrm{Coh}(X)$, there are isomorphisms, functorial in $\mathcal{F}$,

$$\mathrm{Ext}^i_X(\mathcal{F}, \omega_X) \xrightarrow{\sim} H^{D-i}(X, \mathcal{F})^*, \text{ for all } i,$$

where $D = \dim X$.

Proof is to reduce to $\mathbb{P}^N$ case. See [H] § III.7. $\square$.

Again, can extend to some coherent sheaves. I'm not actually sure offhand ~~of~~ the correct generality — I encourage you to try to look it up, since it will give you some experience with searching the literature. Try the Stacks Project.

Here's a statement that's OK for sure.

Suppose $\mathcal{F}, \mathcal{G} \in \mathrm{Coh}(X)$ and $\mathcal{G}$ is perfect, i.e. $\exists$ $0 \to L_n \to L_{n-1} \to \cdots \to L_0 \to \mathcal{G} \to 0$ exact with each L_i locally free of finite rank. Then

$$\mathrm{Ext}^i_X(\mathcal{F}, \mathcal{G} \otimes \omega_X) \cong \mathrm{Ext}^{D-i}_X(\mathcal{G}, \mathcal{F})^*$$

for all i, where $D = \dim X$ (and again X is projective and CM).

<u>Ext^1's</u> $\mathcal{F}', \mathcal{F}'' \in \mathrm{Qcoh}(X)$. An <u>extension</u> or better <u>1-extension</u> of $\mathcal{F}''$ by $\mathcal{F}'$ is a short exact sequence in $\mathrm{Qcoh}(X)$,

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0. \quad (+)$$

An <u>isomorphism</u> of 1-extensions is an isomorphism of short exact sequences that is the identity map on $\mathcal{F}'$ and $\mathcal{F}''$!

Given an extension $(+)$, get boundary map

$$\mathrm{Hom}(\mathcal{F}'', \mathcal{F}'') \xrightarrow{\delta} \mathrm{Ext}^1(\mathcal{F}'', \mathcal{F}').$$

$\delta(\mathrm{id}_{\mathcal{F}''})$ is the extension class of $(+)$.

<u>Prop</u> The above construction yields a bijection between the set of isomorphism classes of 1-extensions and $\mathrm{Ext}^1(\mathcal{F}'', \mathcal{F}')$.

Pf vague sketch. Choose an injective resolution

$$\mathcal{F}' \to I^0 \xrightarrow{\partial^0} I^1 \xrightarrow{\partial^1} I^2 \cdots$$

Given a class in $\mathrm{Ext}^1(\mathcal{F}'', \mathcal{F}')$, choose representative $\mathcal{F}'' \xrightarrow{\varphi} I^1$ with $\partial^1 \circ \varphi = 0$. So $\varphi(\mathcal{F}'') \subseteq \mathrm{Im}(\partial^0)$.

Form pull back

$$\begin{array}{ccccccccc}
0 \to & \mathcal{F}' & \to & I^0 \times_{\mathrm{Im}(\partial^0)} \mathcal{F}'' & \to & \mathcal{F}'' & \to 0 \\
& \| & & \downarrow & & \downarrow & \\
0 \to & \mathcal{F}' & \to & I^0 & \xrightarrow{\partial^0} & \mathrm{Im}(\partial^0) & \to 0
\end{array}$$

to get 1-extension. Check these two constructions are mutually inverse.

Sometimes this information is overly refined: you might prefer to calculate

$$\mathrm{Aut}(\mathcal{F}')\backslash \mathrm{Ext}^1(\mathcal{F}'',\mathcal{F}')/\mathrm{Aut}(\mathcal{F}''),$$

to get set of iso classes of exact sequences

$$0\to\mathcal{G}'\to\mathcal{G}\to\mathcal{G}''\to 0$$

with $\mathcal{G}'\cong\mathcal{F}'$, $\mathcal{G}''\cong\mathcal{F}''$!

BTW, above Prop was really an abelian category statement (if it has enough injectives).

Ex $X=\mathbb{A}^2_k$, $\mathcal{F}''=\mathcal{O}_p$, $p=(0,0)$,
$\mathcal{F}''=I_p$.

Use

$$0\to k[x,y]\xrightarrow{\binom{-y}{x}}k[x,y]^2\xrightarrow{(x\ y)}k[x,y]\to k[x,y]/(x,y)\to 0$$

to get

$$\mathrm{Ext}^i_{k[x,y]}(k[x,y]/(x,y),(x,y))$$

$$=H^i\left(\mathrm{Hom}(k[x,y],(x,y))\xrightarrow{\delta^0}\mathrm{Hom}(k[x,y]^2,(x,y))\xrightarrow{\delta^1}\mathrm{Hom}(k[x,y],(x,y))\right),$$ i.e. of

$$(x,y)\xrightarrow{\binom{x}{y}}(x,y)^2\xrightarrow{(-y\ x)}(x,y).$$

What is H^1? You compute!