Exercise 7.2.1

If $P, Q$ are points in the Klein model, the Euclidean line between them
is a line in the Klein model. Since line
segments are also Euclidean, axioms 1 and 2 of Euclid are immediate.

Exercise 7.2.2

Suppose $P$ is a point in the Klein model: let $RS$ be a chord passing
through $P$.

Then, as a function of the position of a point $Q$, 

\[ d_K(P, Q) = \frac{1}{2} \left| \ln \left( C \cdot \frac{QR}{QS} \right) \right| \]

where $C = \frac{PS}{PR}$ is constant.

Now $QR, QS$ vary continuously in the coordinates (in the Cartesian sense) of $Q$, and
\( \ln(-) \) is continuous, so $d_K(P, Q)$ is continuous in the Cartesian coordinates of $Q$. 

If $Q$ is within (Euclidean) distance $\epsilon$ of $P$, then $QS < \epsilon$, and $QR > RS - \epsilon$, so 
\[
\frac{QR}{QS} > \frac{RS - \epsilon}{\epsilon},
\]
which goes to infinity as $\epsilon \to 0^+$. Hence $d_k(P,Q)$ takes on arbitrarily large values as $Q$ varies.

On the other hand, if the distance from $P$ to $Q$ is $\epsilon$, then $QS > PS - \epsilon$, 
$QR < PR + \epsilon$, and, taking $\epsilon < \min \left\{ \frac{1}{n} PR, \frac{1}{n} PS \right\}$,
we get 
\[
\frac{(PS)(QR)}{(PR)(QS)} < \frac{(PS)(PR + \epsilon)}{(PR)(PS - \epsilon)} < \frac{(PS)(PR + \frac{1}{n} PR)}{(PR)(PS - \frac{1}{n} PS)}
\]
\[
= \frac{(PS)(PR)}{(PR)(PS)} \cdot \frac{1 + \frac{1}{n}}{1 - \frac{1}{n}} = \frac{n + 1}{n - 1} = 1 + \frac{2}{n - 1}.
\]

For $n$ sufficiently large, then,
\[
d_k(P,Q) = \frac{1}{2} \ln \left( \frac{(PS)(QR)}{(PR)(QS)} \right) < \frac{1}{2} \ln \left( 1 + \frac{2}{n - 1} \right)
\]
can be made as close to $\frac{1}{2} \ln(1) = 0$ as we like.

So $d_k(P,Q)$ takes on arbitrarily small values.

Since $d_k(P, -)$ is continuous, the Intermediate Value Theorem implies that $d_k(P, -)$ takes on all nonnegative values.
Now define the circle of radius \( r \) around \( P \) by
\[
C = \{ Q \mid \text{dist}_K(P, Q) = r \}.
\]
By the previous paragraph, this is a nonempty set!

Exercise 7.2.3

If \( L, M \) are parallel, the lines \( N \) perpendicular to \( L \) pass through \( \text{pole}(L) \); those perpendicular to \( M \) pass through \( \text{pole}(M) \). Hence a common perpendicular passes through \( \text{pole}(L) \) and \( \text{pole}(M) \). Such a Euclidean line is unique (unless \( \text{pole}(L) = \text{pole}(M) \), in which case the tangents to the endpoints of \( L, M \) agree, showing that \( L = M \)).

The Euclidean line \( \overline{\text{pole}(L)(\text{pole}(M))} \) will
intersect the Klein disk—since it lies between the tangents to the endpoints of $M$, for example, this follows from Pasch’s Axiom—unless the line equals one of these two tangents, i.e., both poles (of $L$ and $M$) lie on a single line tangent to the circle. But this happens exactly when the chords $L$ and $M$ share a common endpoint: i.e., the lines are limiting parallels of each other.

**Exercise 7.2.4**

Suppose $\angle QPT$ is a right angle. Then $CB$ is perpendicular to $PA$, so $PQ$ passes through the pole of $CB$. But this means $PQ$ passes through two points (Pole of $AB$ and Pole of $CB$) that lie on the same line (the line through $B$ tangent to the circle), so $PQ$ equals that line—and since the line is tangent to the circle, $PQ$ contains no points in the Klein model, a contradiction.
It remains to show that the angle of parallelism cannot be greater than a right angle.

Suppose it is. Construct a line through P perpendicular to PQ. Since the angle on the side of QPT is less than \( \angle QPT \), and \( \overline{CB} \) is a limiting parallel to \( \overline{AB} \), the line \( \ell \) will intersect \( \overline{AB} \). On the other hand, line \( \overline{PQ} \) intersects \( \ell \) and \( \ell' \) both perpendicularly, and Euclid's Prop. 5-28 then implies \( \ell \) and \( \ell' \) are parallel, contradicting the previous sentence. So the angle of parallelism cannot be greater than a right angle.