Math 241, Midterm 2

Name______________________________

Signature__________________________

Circle your section:

BD1  8-8:50 a.m., 147 Altgeld, Dan Lior
BD2  9-9:50 a.m., 347 Altgeld, Dan Lior
BD3  10-10:50 a.m., 443 Altgeld, Daniel Morton
BD4  10-10:50 a.m., 140 Henry, Christopher Lee
BD5  12-12:50 p.m., 143 Henry, Dimitris Koukoulopoulos
BD6  3-3:50 p.m., 154 Henry, Dimitris Koukoulopoulos

Show all work. Justify your answers.

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Problem 1. (12 points) Give an equation for the tangent plane to the surface \( x^2 + 2y^2 + z^2 = 4 \) at the point \((1,1,1)\). Show your work.

Let \( g(x, y, z) = x^2 + 2y^2 + z^2 - 4 \).

Then \( \nabla g(1,1,1) = \langle 2x, 4y, 2z \rangle \).

\( \nabla g(1,1,1) = \langle 2, 4, 2 \rangle \) is normal vector.

This gives equation
\[
\langle x-1, y-1, z-1 \rangle \cdot \langle 2, 4, 2 \rangle = 0 \quad \text{or}
\]
\[
2(x-1) + 4(y-1) + 2(z-1) = 0
\]

Problem 2. (8 points) Suppose \( h(x, y) \) is a function for which
\[
h(1,2) = 4 \quad \text{and} \quad \nabla h(1,2) = \langle 3, 5 \rangle.
\]

Use linear approximation to estimate \( h(1.1, 1.9) \). Show your work.

\[
h(1.1, 1.9) \approx h(1,2) + \nabla h(1,2) \cdot \langle -0.1, -0.1 \rangle
\]

\[
4 + 3 \cdot (-0.1) = 4 + 0.3 - 0.5 = 3.8
\]
Problem 3. (10 + 4 points) Let

\[ f(x, y) = x^2 - \frac{1}{3}y^3 - x^2y + y. \]

(with domain the entire plane).

(1) Find the critical points of \( f(x, y) \). Show your work.

\[ \nabla f = \left< 2x - 2xy, -y^2 - x^2 + 1 \right> \]

\[ \nabla f = 0 \implies 2x = 2xy \quad \text{and} \quad x^2 + y^2 = 1 \]

Cases

\[
\begin{align*}
    x & = 0 & \quad \text{or} & & y & = \frac{1}{2} \\
    0^2 + y^2 & = 1 & & \quad \text{so} & & y & = \pm 1 \\
\end{align*}
\]

\[ \boxed{(0, 1) \quad \text{and} \quad (0, -1)} \]

(2) Compute the Hessian \( H(f)(3, 0) \) of \( f \) at the point (3, 0). Show your work.

\[ H(f) = \begin{pmatrix}
    \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\
    \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2}
\end{pmatrix} = \begin{pmatrix}
    2 - 2y & -2x \\
    -2x & -2y
\end{pmatrix} \]

\[ H(f)(3, 0) = \begin{pmatrix}
    2 & -6 \\
    -6 & 0
\end{pmatrix} \]

Problem 4. (10 points) Let \( f(x, y, z) = x^3yz + 3x(z^2 - 1) + 4z \). Find the unit direction \( u \) for which \( D_u f(1, 0, 0) \) is the largest. Show your work.

\[ \nabla f = \left< 3x^2yz + 3x(z^2 - 1), x^3z, x^3y + 3x(2z) + 4 \right> \]

\[ \nabla f(1, 0, 0) = \left< -3, 0, 4 \right> \]

\[ ||\nabla f(1, 0, 0)|| = \sqrt{(-3)^2 + 4^2} = \sqrt{25} = 5. \]

So, \[ \hat{u} = \frac{\nabla f(1, 0, 0)}{||\nabla f(1, 0, 0)||} = \frac{\left< -3, 0, 4 \right>}{5} = \left< \frac{-3}{5}, 0, \frac{4}{5} \right> \]
Problem 5. (12 points) Suppose \( g(x, y) \) is some function with continuous second partial derivatives (you are not given a formula for \( g \)). Suppose that \( g \) has critical points at \((0, 0), (1, 0)\) and \((0, 3)\). Suppose that the Hessian of \( g \) at the three critical points is:

\[
H(g)(0, 0) = \begin{pmatrix} 0 & 3 \\ 3 & 1 \end{pmatrix}, \quad H(g)(1, 0) = \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}, \quad H(g)(0, 3) = \begin{pmatrix} 5 & 1 \\ 1 & 1 \end{pmatrix}.
\]

Classify each of the critical points \((0, 0), (1, 0)\), and \((0, 3)\) of \( g \) as local max, local min, saddle point, or "not enough information," and show your work/justify your answers very briefly.

\[
\begin{array}{ccc}
(0, 0) & (1, 0) & (0, 3) \\
D = 0 & D = (-2) \cdot (2) - 1 \cdot 1 & D = 5 \cdot 1 - 1 \cdot 1 \\
& = -9 < 0 & = 4 > 0 \\
& \text{saddle point} & \frac{\partial^2 g}{\partial x^2} = 5 > 0 \\
& & \text{local min.}
\end{array}
\]

Problem 6. (12 points) Let

\[
x(u, v) = 2u^2, \quad y(u, v) = 3v - 2u
\]

be functions of the variables \( u \) and \( v \). Suppose \( f(x, y) \) is a function of the variables \( x \) and \( y \) and let

\[
F(u, v) = f(x(u, v), y(u, v))
\]

(the composite function). Suppose also that we have the following tables of values:

<table>
<thead>
<tr>
<th>( (x, y) )</th>
<th>( \frac{\partial f}{\partial x}(x, y) )</th>
<th>( (x, y) )</th>
<th>( \frac{\partial f}{\partial y}(x, y) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (1, 0) )</td>
<td>-3</td>
<td>( (1, 0) )</td>
<td>-2</td>
</tr>
<tr>
<td>( (2, 1) )</td>
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<td>( (2, 1) )</td>
<td>-1</td>
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<tr>
<td>( (2, 3) )</td>
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<td>( (2, 3) )</td>
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</tr>
<tr>
<td>( (3, 2) )</td>
<td>-7</td>
<td>( (3, 2) )</td>
<td>3</td>
</tr>
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</table>

Compute \( \frac{\partial F}{\partial u}(1, 1) \).

Justify your answer.

\[
x(1, 1) = 2(1)^2 = 2, \quad y(1, 1) = 3 \cdot 1 - 2 \cdot 1 = 1.
\]

\[
\left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u} \right\rangle = \left\langle 4u, -2 \right\rangle, \quad \text{so}
\]

\[
\left\langle \frac{\partial x}{\partial u}(1, 1), \frac{\partial y}{\partial u}(1, 1) \right\rangle = \left\langle 4, -2 \right\rangle.
\]

\[
\nabla f(x(1, 1), y(1, 1)) = \nabla f(2, 1) = \langle 4, -1 \rangle.
\]

\[
\nabla f(2, 1) \cdot \left( \frac{\partial x}{\partial u}(1, 1), \frac{\partial y}{\partial u}(1, 1) \right) = \langle 4, -2 \rangle \cdot \langle 4, -1 \rangle = 18.
\]
Problem 7. (14 + 3 + 3 points) Let
\[ f(x, y) = \frac{y}{2} - x^2. \]

(1) Use the method of Lagrange multipliers to find the maximum and minimum of \( f(x, y) \) subject to the constraint \( 16x^2 + y^2 = 64 \). Show your work.

(2) Graph the parabola \( f(x, y) = 0 \) and the ellipse \( 16x^2 + y^2 = 64 \) on the same graph.

(3) Suppose \( f \) has a maximum at the point \((a, b)\), and let \( f(a, b) = M \) be the maximum value. How are the graphs of the ellipse and the level set \( f(x, y) = M \) related at \((a, b)\)?

Let \( g(x, y) = 16x^2 + y^2 - 64 \). Then

\[ \nabla f = \langle -2x, \frac{1}{2} \rangle, \quad \nabla g = \langle 32x, 2y \rangle. \]

Set \( \nabla f = \lambda \nabla g \), get

\[
\begin{align*}
-2x &= 32x \lambda \\
\frac{1}{2} &= 2y \lambda \\
16x^2 + y^2 &= 64
\end{align*}
\]

To solve \(-2x = 32x \lambda\):

**Cases**

\( x = 0 \)

\[ 16 \cdot 0^2 + y^2 = 64 \]

So \( y = \pm 8 \)

Points \((0, 8)\), \((0, -8)\)

\[
\begin{align*}
f(0, 8) &= 4, \\
f(0, -8) &= -4,
\end{align*}
\]

Maximum

\[
\begin{align*}
f(\sqrt{3}, -4) &= -5, \\
f(-\sqrt{3}, -4) &= -5
\end{align*}
\]

Minimum

13) They are tangential at \((a, b)\) — alternatively, they are level sets of functions \( f, g \) whose gradients \( \nabla f(a, b) \) and \( \nabla g(a, b) \) are parallel.
Problem 8. (4 + 4 + 4 points) Let \( g(x, y) = (y - x^2)(y - 2x^2) \).

(1) Let \( r(t) = (at, bt) \) be a parametrized line through the origin (where \( a, b \) are some constants, not both zero). Show that \( g(r(t)) \) has a local minimum at \( t = 0 \) for every choice of \( a \) and \( b \).

\[
G(t) = g(r(t)) = (bt - a^2t^2)(bt - 2a^2t^2) \\
= b^2t^2 - a^2bt^3 - 2a^2bt^3 + 2a^4t^4.
\]

\[
G'(t) = 2b^2t - 2a^2bt^2 + 8a^4t^3 \\
G''(t) = 2b^2 - 18a^2bt + 24a^4t^2.
\]

\begin{align*}
\text{Cases} & \quad \text{If } b \neq 0 \quad \begin{cases}
G'(0) = 0, \\
G''(0) > 0,
\end{cases} \\
& \quad \text{so, local min at } t = 0.
\end{align*}

\begin{align*}
\text{If } b = 0 & \quad G'(t) = 2a^4t^4, \\
& \quad \text{if } 2a^4 > 0, \\
& \quad \text{then } G'(t) \text{ has a local minimum at } t = 0.
\end{align*}

(2) Graph the level set \( g(x, y) = 0 \) in the plane. On the same graph, label the regions where \( g(x, y) \) is positive (with a +) and where \( g(x, y) \) is negative (with a −).

(3) Is \((0, 0)\) a local maximum, local minimum, or neither for \( g(x, y) \)? Justify your answer. ["Not enough information" will be worth zero points.] \textbf{Hint: look at your picture from part (2).}

The regions labelled "+" and "−" both have \((0, 0)\) on their boundaries, so \( g(x, y) \) has neither a local max nor a local min at \((0, 0)\).