Problem 1. (4 points) Suppose \( \mathbf{v} = (v_1, v_2, v_3) \) and \( \mathbf{w} = (w_1, w_2, w_3) \) are two vectors. Give the formula for the dot product \( \mathbf{v} \cdot \mathbf{w} \).

\[ \mathbf{v}_1 \mathbf{w}_1 + \mathbf{v}_2 \mathbf{w}_2 + \mathbf{v}_3 \mathbf{w}_3 \]

Problem 2. (6 points) Let \( \mathbf{v} = (1, 2, -3) \) and \( \mathbf{w} = (1, 4, -5) \). Find a (nonzero) vector perpendicular to \( \mathbf{v} \) and \( \mathbf{w} \). Show your work.

\[
\mathbf{v} \times \mathbf{w} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
1 & 2 & -3 \\
1 & 4 & -5 \\
\end{vmatrix} = \mathbf{i} \begin{vmatrix} 2 & -3 \\
4 & -5 \\
\end{vmatrix} - \mathbf{j} \begin{vmatrix} 1 & -3 \\
1 & -5 \\
\end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & 2 \\
1 & 4 \\
\end{vmatrix}
\]

\[= \langle -10, -(-12), -(-5--3), 4-2 \rangle = \boxed{\langle 2, 2, 2 \rangle} \]

Problem 3. (6 points) Find a linear equation of the plane containing the point \( (2, -3, 4) \) and having normal vector \( (2, 1, 3) \). Show your work.

\[ 0 = \langle x-2, y-(-3), z-4 \rangle \cdot \langle 2, 1, 3 \rangle \]

\[= 2(x-2) + (y+3) + 3(z-4) \]

\[= 2x-4 + y+3 + 3z-12 \]

\[2x+y+3z = 13 \]

\boxed{2x+y+3z = 13}
Problem 4. (5 points) Let $a$, $b$, $c$, and $d$ be nonzero vectors. Circle all (and only!) the expressions that make sense:

(i) $a \times (b \cdot c)$
(ii) $a + b \times c$
(iii) $\frac{\|a\|}{a}$
(iv) $(a \times (b \times c)) \cdot d$

Problem 5. (10 + 6 points) Give a parametrization (either a vector-valued function or parametric equations) for the line through the two points $p = (1, -1, 2)$ and $q = (2, 3, 3)$. Show your work.

\[
\overrightarrow{pq} = \langle 2, 3, 3 \rangle - \langle 1, -1, 2 \rangle = \langle 1, 4, 1 \rangle.
\]

\[
\vec{r}(t) = \langle 1, -1, 2 \rangle + t \langle 1, 4, 1 \rangle
\]

or
\[
\vec{r}(t) = \langle 1 + t, -1 + 4t, 2 + t \rangle
\]

Is the line above perpendicular to the line given by $\gamma(t) = (1 - 4t, -1 + t, 2 + t)$? Justify your answer.

Lines point in directions $\langle 1, 4, 1 \rangle$ and $\langle -4, 1, 1 \rangle$ respectively.

$\langle 1, 4, -1 \rangle \cdot \langle -4, 1, 1 \rangle = -4 + 4 - 1 = -1$,

so not perpendicular.
Problem 6. (10 + 6 points) Let $a$ be an unknown constant.

(1) Compute the volume of the parallelepiped with adjacent edges given by the vectors

\[
\mathbf{u} = (1, 2, 0), \quad \mathbf{v} = (2, 0, 3), \quad \mathbf{w} = (a, 1, -1)
\]

(you will get an expression that depends on $a$). Show your work.

\[
\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix}
1 & 2 & 0 \\
2 & 0 & 3 \\
a & 1 & -1
\end{vmatrix}
\]

\[
= 1 \begin{vmatrix}
0 & 3 \\
-2 & a
\end{vmatrix} - 2 \begin{vmatrix}
2 & 3 \\
a & -1
\end{vmatrix} + 0 \begin{vmatrix}
2 & 0 \\
a & 1
\end{vmatrix}
\]

\[
= -3 - 2(-2-3a) = -3 + 4 + 6a = 1 + 6a.
\]

So volume $\boxed{|1 + 6a|}$

(2) For which values of $a$, if any, are the three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ coplanar? Justify your answer.

The vectors are coplanar if and only if the volume is zero, i.e., if and only if

\[1 + 6a = 0, \quad \text{i.e.,} \quad \boxed{a = -\frac{1}{6}}\]

Problem 7. (6 points) Suppose $a$, $b$ and $c$ are nonzero vectors. If $a \times b = a \times c$, must it be true that $b = c$? If it is true, explain why; if it is false, give a counterexample.

This is false. For example, take $\mathbf{a} = \langle 1, 0, 0 \rangle$, $\mathbf{b} = \langle 1, 0, 0 \rangle$, $\mathbf{c} = \langle 2, 0, 0 \rangle$. Then

\[
\mathbf{a} \times \mathbf{b} = \mathbf{0} = \mathbf{a} \times \mathbf{c} \quad \text{but} \quad \mathbf{b} \neq \mathbf{c}.
\]
Problem 8. (16 points) A helicopter has instruments that display its velocity and acceleration (but not position). According to the instruments, the helicopter's acceleration is described by the function

\[ \mathbf{a}(t) = (2e^t + 2, 3 \cos(t), 2). \]

Suppose that its position and velocity at time \( t = 0 \) are

\[ \mathbf{r}(0) = (3, -3, -1), \quad \mathbf{v}(0) = (2, 2, 1). \]

Compute the function \( \mathbf{r}(t) \) that gives the helicopter's position at time \( t \). Show your work.

\[
\mathbf{v}(t) = \left( 2e^t + 2t + c_1, 3 \sin(t) + c_2, 2t + c_3 \right)
\]

where \( c_1, c_2, c_3 \) are constants. Now

\[
\mathbf{v}(0) = \left( 2 + 0 + c_1, 3 \sin(0) + c_2, c_3 \right) = (2, 2, 1)
\]

so \( \langle c_1, c_2, c_3 \rangle = (0, 2, 1) \).

\[
\mathbf{v}(t) = \left( 2e^t + 2t, 3 \sin(t) + 2, 2t + 1 \right).
\]

\[
\mathbf{r}(t) = \left( \int (2e^t + 2t) \, dt, \int (3 \sin(t) + 2) \, dt, \int (2t + 1) \, dt \right)
\]

\[
= \left( 2e^t + t^2 + b_1, -3 \cos(t) + 2t + b_2, 2t + 2b_3 \right)
\]

where \( b_1, b_2, b_3 \) are constants. Now

\[
\mathbf{r}(0) = \left( 2 + 0 + b_1, -3 \cos(0) + 0 + b_2, b_3 \right) = (3, -3, -1)
\]

so \( \langle b_1, b_2, b_3 \rangle = (1, 0, -1) \).

So

\[
\mathbf{r}(t) = \left( 2e^t + t^2 + 1, -3 \cos(t) + 2t, 2t + 2 - 1 \right).
\]
Problem 9. (6+6+6 points)

1. Compute the velocity function \( v(t) \) of the vector-valued function 
   \[ r(t) = \langle t^2, t^3 - 3t^2 - 3t \rangle. \]

   Show your work.

   \[ \dot{r}(1) = \langle 2t, 3t^2 - 6t - 3 \rangle. \]

2. Compute the unit tangent vector at time \( t = -1 \). Show your work.

   \[ \dot{r}'(-1) = \langle -2, 3(-1)^2 - 6(-1) - 3 \rangle = \langle -2, 6 \rangle. \]

   \[ \| \dot{r}'(-1) \| = \sqrt{4 + 36} = \sqrt{40} = 2\sqrt{10}. \]

   \[ \frac{\dot{r}}{\| \dot{r} \|} = \frac{1}{2\sqrt{10}} \langle -2, 6 \rangle = \left\langle \frac{-2}{\sqrt{10}}, \frac{6}{\sqrt{10}} \right\rangle. \]

3. At what time(s), if any, does the velocity vector point in the same direction as the vector \( \langle 1, 1 \rangle \)? Show your work.

   Need \( \dot{r}'(t) = \lambda \cdot \langle 1, 1 \rangle \) for some \( \lambda > 0 \).

   Get \( 2t = \lambda = 3t^2 - 6t - 3 \) or \( 3t^2 - 9t - 3 = 0 \).

   \( (3t + 1)(t - 3) = 0 \)

   \[ \dot{r}'(3) = \langle 6, 27 - 18 - 3 \rangle = \langle 6, 6 \rangle \]

   \[ \dot{r}'(-\frac{1}{3}) = \langle -\frac{2}{3}, \frac{1}{3} + 2 - 3 \rangle = \langle \frac{2}{3}, -\frac{2}{3} \rangle. \]

   \( t = 3 \) same direction \( \frac{\dot{r}}{\| \dot{r} \|} \) \( \langle -\frac{1}{3} \rangle \) opposite direction

Problem 10. (6 points) Show that the curve given by the vector-valued function

\[ r(t) = \langle \sin(t)\sqrt{2}, 2 - \cos(t), 1 + \cos(t) \rangle \]

lies on the sphere of radius 2 centered at the point \( (0, 1, 0) \). Justify your answer.

sphere has equation \( x^2 + (y-1)^2 + z^2 = 4 \).

Substitute, get

\[ (\sin(t)\sqrt{2})^2 + (2 - \cos(t) - 1)^2 + (1 + \cos(t))^2 \]

\[ = 2\sin^2 t + 1 - 2\cos t + \cos^2 t + 1 + 2\cos t + \cos^2 t \]

\[ = 2 + 2\sin^2 t + 2\cos^2 t = 2 + 2 = 4, \) so every output \( r(t) \) satisfies the equation of the sphere.