Reminder: Last time, saw that flux
of an inverse-square vector field
\[ \mathbf{F} = \frac{c \mathbf{r}^2}{|\mathbf{r}|^3} \quad (c \text{ constant}) \]
is the same over every closed surface
enclosing \((0,0,0)\). Gauss's Law.

Today

1) When is a vector field \( \mathbf{F} \)
the curl of another vector field \( \mathbf{G} \)?

2) Stokes's Theorem.
Fact: If \( \vec{F} = \text{curl}(\vec{G}) = \nabla \times \vec{G} \)

then \( \text{div}(\vec{F}) = 0 \).

Pf. Calculate:

\[
\text{curl}(\vec{G}) = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
G_1 & G_2 & G_3
\end{vmatrix}
\]

\[
= \left\langle \frac{\partial G_3}{\partial y} - \frac{\partial G_2}{\partial z}, -\left( \frac{\partial G_3}{\partial x} - \frac{\partial G_1}{\partial z} \right), \frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y} \right\rangle
\]

Then

\[
\text{div}(\text{curl}(\vec{G}))
\]

\[
= \frac{\partial^2 G_3}{\partial x \partial y} - \frac{\partial^2 G_2}{\partial x \partial z} - \frac{\partial^2 G_3}{\partial y \partial x} + \frac{\partial^2 G}{\partial y \partial z} + \frac{\partial^2 G_2}{\partial x \partial y} - \frac{\partial^2 G_3}{\partial z \partial x}
\]

\[= 0.\]

So, if \( \text{div}(\vec{F}) \neq 0 \) then \( \vec{F} \) is not the curl of any vector field.
Suppose \( \text{div}(\vec{F}) = 0 \).

Is there a vector field \( \vec{G} \) such that
\[
\text{curl}(\vec{G}) = \vec{F}.
\]

A. Yes, provided some condition holds on the domain of \( \vec{F} \). For example, if \( \vec{F} \) has continuous derivatives everywhere, that’s good enough.
We want to know $\hat{G}$.

Let $\mathbf{F} = \langle F_1(x, y, z), F_2(x, y, z), F_3(x, y, z) \rangle$.

\[ \hat{G}(x, y, z) = \left\langle 0, \int_0^x F_3(u, y, z) \, du - \int_0^z F_1(0, y, z) \, dv, \int_0^x F_2(u, y, z) \, du \right\rangle \]

Proof that this works: we want to show that $\text{curl}(\hat{G}) = \mathbf{F}$. We'll use

\[ \text{curl}(\hat{G}) = \left\langle \frac{\partial G_3}{\partial y} - \frac{\partial G_2}{\partial z}, -\left( \frac{\partial G_3}{\partial x} - \frac{\partial G_1}{\partial z} \right), \frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y} \right\rangle \]

We get

\[ \text{curl}(\hat{G}) = \langle \frac{\partial }{\partial y} \left( -\int_0^x F_2(u, y, z) \, du \right) - \frac{\partial }{\partial z} \left( \int_0^x F_3(u, y, z) \, du - \int_0^z F_1(0, y, z) \, dv \right), \]

\[ -\frac{\partial }{\partial x} \left( -\int_0^x F_2(u, y, z) \, du \right), \frac{\partial }{\partial x} \left( \int_0^x F_3(u, y, z) \, du - \int_0^z F_1(0, y, z) \, dv \right) \rangle \]

Using the Fundamental Theorem of Calculus,
\[ \frac{\partial}{\partial x} \left( \int_0^x F_2(u,y,z) \, du \right) = F_2(x,y,z), \]
\[ \frac{\partial}{\partial y} \left( \int_0^x F_3(u,y,z) \, du \right) = F_3(x,y,z). \] Also,
\[ \frac{\partial}{\partial z} \int_0^z F_1(v,y,z) \, dv = 0 \] since the expression being differentiated is constant as far as \( x \) is concerned.

Finally,
\[ \frac{\partial^2}{\partial y^2} \left( -\int_0^x F_2(u,y,z) \, du \right) - \frac{\partial^2}{\partial z^2} \left( \int_0^x F_3(u,y,z) \, du \right) + \frac{\partial}{\partial z} \int_0^z F_1(v,y,z) \, dv \]
\[ = \int_0^x \left[ -\frac{\partial F_2}{\partial y}(u,y,z) - \frac{\partial F_3}{\partial z}(u,y,z) \right] \, du + F_1(0,y,z). \]

Now, since \( \text{div}(\mathbf{F}) = 0 \), we have
\[ \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 0, \] or
\[ \frac{\partial F_1}{\partial x} = -\frac{\partial F_2}{\partial y} - \frac{\partial F_3}{\partial z}. \]
Substituting into here, we get
\[ = \int_0^x \frac{\partial F_1}{\partial x}(u,y,z) \, du + F_1(0,y,z). \]
= \mathbf{F}(x, y, z) - \mathbf{F}(0, y, z) + \mathbf{F}(0, y, z)
= \mathbf{F}(x, y, z) \text{ by the Fundamental Theorem of Calculus.}

We've now calculated all three components in the boxed formula for \( \text{curl} (\mathbf{G}) \), and found that \( \text{curl} (\mathbf{G}) = \mathbf{B} \), which is what we wanted!

To summarize, suppose \( \mathbf{F} = (F_1, F_2, F_3) \) is a vector field that has continuous derivatives everywhere. If \( \text{div}(\mathbf{F}) = 0 \), then \( \mathbf{F} = \text{curl}(\mathbf{G}) \) where \( \mathbf{G}(x, y, z) \) is given by the following formula:
\[ \vec{G}(x,y,z) = \begin{pmatrix} 0, \int_0^x f_3(u,y,z) \, du - \int_0^z f_1(u,y,z) \, dv, -\int_0^x f_2(u,y,z) \, du \end{pmatrix} \]

Example

\[ \vec{F}(x,y,z) = \begin{pmatrix} yz^2 + y, 2z - x^2, 1 \end{pmatrix} \]

div \vec{F} = 0 + 0 + 0 = 0. So it is a curl.

Let

\[ \vec{g} = \begin{pmatrix} 0, \int_0^x 1 \, du - \int_0^z (yz^2 + y) \, dv, -\int_0^x (2z - u^2) \, du \end{pmatrix} \]

\[ = \begin{pmatrix} 0, x - [y \frac{v^3}{3} + yv]_0^z, -2zu + \frac{u^3}{3} \end{pmatrix} \]

\[ = \begin{pmatrix} 0, x - y \frac{z^3}{3} - yz, -2z x + \frac{x^3}{3} \end{pmatrix} \]

Claim was that curl(\vec{g}) = -\vec{F}.

let's check:
\( \text{curl}(\mathbf{A}) = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & x - \frac{y^2}{3} - yz & -2zx + \frac{x^3}{3} \end{vmatrix} \)

\[
= \langle 0, 0, 0 \rangle - \langle -yz^2 - y, -2z + x^2, 1 \rangle,
\]

\[
= \langle yz^2 + y, 2z - x^2, 1 \rangle. \quad \checkmark
\]

Application of this formula to compute magnetic flux... later.
Stokes's Theorem "Green's Theorem in Space"

Recall Green's:

\[ \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \, dA \]

Think \( \mathbf{F} = \langle F_1, F_2, 0 \rangle \)

\[ \text{curl} \mathbf{F} = \langle 0, 0, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \rangle \]
Stokes' Theorem generalizes this to

\[ \int_{\partial S} \vec{F} \cdot d\vec{r} = \int_S \text{curl}(\vec{F}) \cdot \hat{n} \, dS. \]
\( \text{curl}(\vec{F}) \cdot \hat{n} \) computes the amount of the spin that \( \vec{F} \) is trying to create that's actually felt in the surface.
Example let 
\[ \mathbf{F} = \langle 2x, 2y, z^3 \rangle, \]

\[ S : \quad z = 4 - x^2 - y^2, \]

let's compute 
\[ \int_S (\mathbf{F} \cdot d\mathbf{r}) \] and 
\[ \iint_S \text{curl}(\mathbf{F}) \cdot \mathbf{n} \, dS \]

... next time!