Recall

**Green's Theorem**

\[ \mathbf{F}(x,y) = \langle F_1(x,y), F_2(x,y) \rangle \]

Then

\[ \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \, dA \]

For a simple closed curve, positively oriented.

Meaning of \( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \) is curl of \( \mathbf{F} \),

measures extent to which \( \mathbf{F} \) is rotating around each point.

Example (1): \( \mathbf{F}(x,y) = \langle -y, x \rangle \).

\[ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = \frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y) = 2 \text{, clockwise spin.} \]
(2) $\vec{F}(x,y) = \langle y, -x \rangle$.

\[ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = \frac{\partial}{\partial x} (-x) - \frac{\partial}{\partial y} (y) = -2. \]

(clockwise spin)

(3) $R(x,y) = \langle 0, F_2(x,y) \rangle$.
\( \text{can show} \quad \text{Set} \)
\[
\text{curl}(\mathbf{F}) = \lim_{r \to 0} \frac{1}{\pi r^2} \int_{C_r} \mathbf{F} \cdot d\mathbf{r}
\]

where
\[
C_r : \quad \mathbf{r}(t) = (x, y) + (r \cos t, r \sin t) \quad \text{with} \quad 0 \leq t \leq 2\pi.
\]

Rotation per unit area,
\[
\int_{C_r} \mathbf{F} \cdot d\mathbf{r}
\]
measures how much \( F \) rotates circle around its center.
limit is "infinitesimal rotation per unit area!"

Fact: if I calculate this using linear approximation

I get

\[ \text{curl}(\vec{F}) = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}. \]

Ex: Use Green to compute

\[ \int_C \vec{F} \cdot d\vec{r}, \quad \vec{F} = \langle e^{x^2}y, e^{x^2}, y^2 \rangle, \]

C: \( y = 1-x^2 \)

C: \( y = 0 \)
\[ \iint_R \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \, dx \, dy \]

\[ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 2e^{2x} - (-1) = 2e^{2x} + 1. \]

\[ R = \{(x, y) \mid 0 \leq y \leq 1 - x^2 \} \]

Integral is

\[ \int_{-1}^{1} \int_{0}^{1-x^2} (2e^{2x} + 1) \, dy \, dx \]

\[ \int_{-1}^{1} \int_{0}^{1-x^2} (2e^{2x} + 1) (1-x^2) \, dx \]

\[ \int_{-1}^{1} (2e^{2x} + 1) (1-x^2) \, dx \]
\[
\frac{1}{2} \left( 2e^{2x} + 1 \right) - x^2 \int_{-1}^{1} (-x^2)2e^{2x} \, dx
\]

\[
= \left( e^{2x} + x - \frac{x^3}{3} \right) \bigg|_{-1}^{1} + \int_{-1}^{1} (-x^2)2e^{2x} \, dx
\]

\[
= e^2 - e^{-2} + (1 - (-1)) - \left( \frac{1}{3} - \frac{-1}{3} \right)
\]

\[
+ \int_{-1}^{1} (-x^2)2e^{2x} \, dx
\]

\[
= e^2 - e^{-2} + \frac{4}{3} + \int_{-1}^{1} (-x^2)2e^{2x} \, dx
\]

\[
= \left[ \frac{4}{3} + \frac{1}{2}e^2 + \frac{3}{2}e^{-2} \right] \]

integrate by parts twice
Computing area via line integrals!

\[
\begin{array}{c|ccc}
\mathbb{R}^2 & \langle 0, x \rangle & \langle -\frac{1}{2}y, \frac{1}{2}x \rangle & \langle -y, 0 \rangle \\
\frac{\partial F_2}{\partial x} \mid_{\langle 0, x \rangle} & 1 & \frac{1}{2} - \left(\frac{-1}{2}\right) \frac{1}{2} & 1 \\
\frac{\partial F_1}{\partial y} \mid_{\langle -y, 0 \rangle} & 1 & 1 & 1 \\
\end{array}
\]

Now

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_R 1 \, dA = \text{area}(R),
\]

in all three cases
In text:

\[ \int_{C} x \, dy \quad \int_{C} \left( -\frac{1}{2} y \, dx + \frac{1}{2} x \, dy \right) \quad \int_{C} -y \, dx \]
Folium of Descartes
\[ x^3 + y^3 - 3xy = 0 \]

How to compute area \( \mathcal{R} \)? parametrize curve:

Substitute \( y = tx \) in \( x^3 + y^3 = 3xy \),
\[ x^3 + (tx)^3 = 3x(tx) \]
\[ (1+t^3)x^3 = 3tx^2 \] or
\[ (1+t^3)x^3 = 3tx^2 \] or
\[ x = \frac{3t}{1+t^3}, \quad y = \frac{3t^2}{1+t^3} \]

\[ 0 \leq t < \infty. \]

Let's use \( F = \langle -\frac{1}{2}y, \frac{1}{2}x \rangle \) to compute area via:

\[ \int_C F \cdot d\mathbf{r} \]

\( F(t) = \left\langle \frac{(1+t^3)3 - 3t(3t^2)}{(1+t^3)^2}, \frac{(1+t^3)(6t)-3t^2(3t^2)}{(1+t^3)^2} \right\rangle \)

\( F(-1) = \left\langle -\frac{3}{2}, \frac{3}{2} \right\rangle \).
\[ \vec{P} \cdot d\vec{r} = \ldots = \frac{9}{2} \frac{t^2}{(1+t^3)^2}. \]

Now, area is
\[
\int_C \vec{P} \cdot d\vec{r} = \int_0^\infty \frac{9}{2} \frac{t^2}{(1+t^3)^2} \, dt.
\]

Use \( u = 1+t^3 \), \( du = 3t^2 \, dt \).

\[
\int_C \vec{P} \cdot d\vec{r} = \int_0^\infty \frac{9}{2} \frac{3t^2}{(1+t^3)^2} \, dt = \int \frac{3}{2} \frac{du}{u^2}.
\]

\[
\int \frac{3}{2} \frac{du}{u^2} = \left. -\frac{3}{2} u^{-1} \right|_0^\infty = -\frac{3}{2} (1+t^3)^{-1} \bigg|_0^\infty
\]

\[
-\frac{3}{2} (1+t^3)^{-1} = \frac{3}{2} \left( \frac{1}{1+0} \right)^{-1} = \frac{3}{2} \quad \text{area of Folium of Descartes}
\]