\[ E_x \ F^2(x,y) = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle = \left\langle F_1, F_2 \right\rangle \]

\((x,y) \neq (0,0)\).

Q: Could it be conservative?

1. \( \frac{\partial F_2}{\partial x} \) vs. \( \frac{\partial F_1}{\partial y} \)

\[
\frac{\partial F_2}{\partial x} = \frac{y^2 - x^2}{(x^2+y^2)^2} = \frac{\partial F_1}{\partial y} \quad \checkmark
\]

So it could be conservative.

2. Integrate around a loop:

\[ F(t) = \langle \cos t, \sin t \rangle \]

\( \vec{F}(\vec{F}(t)) \cdot \vec{F}(t) > 0 \). (in fact = 1).

In this example
\[ \int_C \overrightarrow{F} \cdot d\overrightarrow{r} = \int_0^{2\pi} 1 \, dt = \left[ 2\pi \right] \neq 0. \]

Vector field is not conservative!

**Fact.** The "hole" at (0,0) is the only thing that could prevent \(\overrightarrow{F}\) from being conservative.

Thus, if \(D \subset \mathbb{R}^2\) is the domain of the vector field \(\overrightarrow{F}(x, y)\) and \(D\) is both path-connected and \(D\) "has no holes," i.e., \(D\) is simply-connected, then \(\overrightarrow{F}\) is conservative if and only if

\[ \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}. \]
simply-connected

not simply-connected
The "hole in the middle" is a manifestation of mathematical notion of topology. It may seem esoteric, but it's not.
§14.4 Green's Theorem

Another conservation law, i.e., version of the Fundamental Theorem of Calculus.

Example

\[ \vec{P}(t) = \langle \cos t, \sin t \rangle, \quad 0 \leq t \leq 2\pi \]

\[ F(x, y) = \langle x^2 - y, y^2 \rangle = \langle F_1, F_2 \rangle, \]

\[ \vec{F}(P(t)) = \langle \cos^2 t - \sin t, \sin^2 t \rangle. \]
\[ \mathbf{r}'(t) = \langle -\sin t, \cos t \rangle. \]
\[ \mathbf{F}(\mathbf{r}'(t)) \cdot \mathbf{r}'(t) = -\sin t \cos^2 t + \sin^2 t + \cos^2 t = 1. \]

Then
\[ \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \left( -\sin t \cos^2 t + \sin^2 t + \cos^2 t \right) dt \]
\[ = \cos^3 t \int_0^{2\pi} \frac{1}{3} + \int_0^{2\pi} \sin^2 t dt + \frac{\sin^3 t}{3} \bigg|_0^{2\pi} \]
\[ = \frac{1}{3} - \frac{1}{3} + \int_0^{2\pi} \left( \frac{1}{2} - \frac{1}{2} \cos 2t \right) dt + 0 \]
\[ = \left( \frac{1}{2} t - \frac{1}{4} \sin 2t \right) \bigg|_0^{2\pi} = \pi \]
\[ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 0 - (-1) = 1. \]

\[ \int_R 1 \, dA = \int_{\text{area of } R} 1. \]

\textbf{Green's Theorem}

\[ \oint_C \mathbf{F} \cdot d\mathbf{r} = \int_C \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \, dA \]
Two hypotheses:

* loop shouldn’t cross itself (simple closed curve)

* region $R$ it encloses should be “on the left” of the loop (curve $C$ is positively oriented).

Ex. not good
Different notation in text; they write
\[ \oint_{c} F_1 \, dx + F_2 \, dy \] for LHS.

We already discussed this notation.
Suppose \( \vec{F}(x, y) = \langle 0, F_2(x, y) \rangle \).

Green says

\[
\oint_C \vec{F} \cdot d\vec{r} = \iint_R \frac{\partial F_2}{\partial x} \, dA
\]

\[
\left( \frac{\partial F_2}{\partial y} = 0 \right).
\]
\[\int_{C} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r}\]

because \(\mathbf{F}\) is vertical only.

(velocity along \(C\) is horizontal along horizontal sides of \(C\), so \(\mathbf{F} \cdot d\mathbf{r} = 0\) there).