Conservative vector fields

\[ \mathbb{E} \quad \vec{F}(x, y) = \langle 2x, 3y^2 \rangle. \]

\[ C_1 : \quad \vec{r}(t) = t \langle 1, 2 \rangle, \quad 0 \leq t \leq 1. \]

\[ \vec{r}'(t) = \langle 1, 2 \rangle. \]

\[ \vec{F}(\vec{r}(t)) = \langle 2t, 3(2t)^2 \rangle. \]

\[ \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) = \langle 2t, 12t^2 \rangle \cdot \langle 1, 2 \rangle \]

\[ = 2t + 24t^2. \]

\[ \int_{C_1} \vec{F} \cdot d\vec{r} = \int_0^1 \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt \]

\[ = \left. \int_0^1 (2t + 24t^2) \, dt = \frac{t^2}{2} + 8t^3 \right|_0^1 = 9. \]
\( C_2 : \overrightarrow{F}(t) = \langle t, 2t^2 \rangle, \ 0 \leq t \leq 1. \)
\( \overrightarrow{F}'(t) = \langle 1, 4t \rangle. \)
\( \overrightarrow{F}(t) \cdot \overrightarrow{F}'(t) = \langle 2t, 3(2t^3) \rangle \cdot \langle 1, 4t \rangle \)
\[ = 2t + 24t^4. \]
\[ \int_{C_2} \overrightarrow{F} \cdot d\vec{r} = \int_0^1 (2t + 24t^4) \, dt \]
\[ = \left[ t^2 + 8t^5 \right]_0^1 = 9. \]

This is because \( \overrightarrow{F} \) is a
radient — integrating
\[ \int_C \vec{F} \cdot d\vec{r} \] measures how much energy the field \( \vec{F} \) puts into a particle moving along path \( C \). But if \( \vec{F} = \nabla f \) is a potential function, that "energy put in" is change in potential energy. So, if the change in potential energy is

\[ f(x_2, y_2) - f(x_1, y_1) \]
Conclusion. The integral \( \int_{C} \vec{F} \cdot d\vec{r} \) only depends on the endpoints of \( C \) provided \( \vec{F} = \nabla \phi \) is a conservative vector field.
Def: A region $D \subseteq \mathbb{R}^2$ is path-connected if every pair of points in $D$ can be connected by a piecewise smooth curve.
Then \( \mathbf{F}(x, y) = \langle F_1, F_2 \rangle \)
be a vector field that is continuous on a path-connected region \( D \subset \mathbb{R}^2 \). Then the line integral \( \int_C \mathbf{F} \cdot d\mathbf{r} \) is independent of path, i.e., is completely determined by the beginning and ending points of the path, if and only if the vector field is conservative on \( D \).
 FTC in one variable

\[ \int_{a}^{b} f'(x) \, dx = f(b) - f(a). \]

Fundamental Theorem of Calculus for Line Integrals

Suppose \( \overrightarrow{F}(x,y) = \nabla f(x,y) \).

Suppose \( \overrightarrow{F} \) is continuous on the open path-connected region \( D \subset \mathbb{R}^2 \) and \( C \) is a piecewise \( C^1 \) curve in \( D \) connecting \((x_1, y_1)\) to \((x_2, y_2)\).
\[ \int_{C} \mathbf{F} \cdot d\mathbf{r} = f(x, y, z) - F(x, y, z) \]

\[ = \int_{C} \mathbf{F} \cdot d\mathbf{r} \]

Diagram: A closed path with points labeled (x1, y1, z1) and (x2, y2, z2) connected by a smooth curve.
Suppose I take
\[ F(x, y) = \nabla A(x, y), \] where
\[ A(x, y) = xy. \] So \[ F(x, y) = \langle y, x \rangle. \]

Integrate around circle:

\[ \vec{r}(t) = \langle \cos t, \sin t \rangle, \quad 0 \leq t \leq 2\pi. \]
\[ \vec{r}'(t) = \langle -\sin t, \cos t \rangle. \]
\[ F(\vec{r}(t)) = \langle \sin t, \cos t \rangle. \]
\[ F(\vec{r}(t)) \cdot \vec{r}'(t) = \cos^2 t - \sin^2 t. \]
\[ = \cos 2t. \]
Could you have predicted? Yes.

Could have calculated integral with \( F(t) = \langle 1, 0 \rangle \) instead, yet same answer because \( F \) is conservative.

If \( F \) is conservative,
\[ \int_{C_2 \cup (-C_1)} \mathbf{F} \cdot d\mathbf{r} = 0. \]

Conversely, if every integral \( \int_C \mathbf{F} \cdot d\mathbf{r} \) around a closed curve is 0, \( \mathbf{F} \) is conservative.

**Summary**

Then suppose \( \mathbf{F} \) is continuous on open, path-connected region \( D \subseteq \mathbb{R}^2 \).

Then \( \mathbf{F} \) is conservative if
\[ \int_C \mathbf{F} \cdot d\mathbf{r} = 0 \]
for any piecewise \( C^1 \) closed curve \( C \) in \( D \).
closed curve: same beginning & ending points, it's a loop.

\[ F(x, y) = \langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \rangle = \langle F_1, F_2 \rangle \]

Could it be conservative?

Exercise
\[ \frac{\partial F_2}{\partial x} = \frac{y^2-x^2}{(x^2+y^2)^2} = \frac{\partial F_1}{\partial y}. \]

So it could be.
Integrate around a loop:
\[ \vec{r}(t) = \langle \cos t, \sin t \rangle, \quad 0 \leq t \leq 2\pi \]
\[ \vec{r}'(t) = \langle -\sin t, \cos t \rangle. \]
\[ \vec{r}(t) \cdot \vec{r}'(t) = \langle -\sin t, \cos t \rangle \cdot \langle -\sin t, \cos t \rangle \]
\[ = \sin^2 t + \cos^2 t = 1. \]
\[ \int_C \vec{F} \cdot d\vec{r} = 2\pi, \quad \neq 0. \]