Last time

Vector field: fn. $\mathbb{R}^2$ from (a subset of) $\mathbb{R}^2$ to $\mathbb{R}^2$ or from a subset of $\mathbb{R}^3$ to $\mathbb{R}^3$.

Sources:
- Velocity fields (wind)
- Force field

Force fields often come from potentials $U$:

$U \rightarrow -\nabla U$

($\nabla F$ is a vector field for a scalar-valued $F$).

[Be aware the $-$ sign]

We say $\vec{F}(x,y,z)$ is conservative if $\vec{F}(x,y,z) = \nabla f(x,y,z)$ for some function $f$, which is the potential.
Useful to know when a v.f. $\mathbf{F}$ is conservative.

**Two observations**

1) If $\mathbf{F}_{\text{xy}} = \langle \frac{\partial u}{\partial x}, \frac{\partial v}{\partial y} \rangle$

   $\qquad = \langle F_1, F_2 \rangle$.

   then

   $\frac{\partial F_1}{\partial y} = \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y} = \left[ \frac{\partial F_2}{\partial x} \right].$

   $\operatorname{Ex} F(x, y) = \langle 2, y \rangle.$

   $\frac{\partial F_2}{\partial x} = 0 \quad \frac{\partial F_1}{\partial y} = 0$

   So could be conservative!
\[ F(x, y) = \langle y, 1 \rangle, \]
\[ \frac{\partial G_1}{\partial y} = 1, \quad \frac{\partial G_2}{\partial x} = 0, \]
so can't be conservative!

(2) If \( F = \nabla U \), can find \( U \) by integrating:

\[ \oint F \cdot d\mathbf{r} = \left< \frac{\partial U}{\partial x}, \frac{\partial U}{\partial y} \right> = \langle F_1, F_2 \rangle \]

So

\[ U = \int \frac{\partial U}{\partial x} \, dx \]

\[ = \int \frac{\partial U}{\partial y} \, dy \]
\[ \int F_1 \, dx = U_1 + C_2(y) \]
\[ \int F_2 \, dy = U_2 + C_1(x) \]

Adjust so these are equal.

\[ F(x, y) = \langle 2, y \rangle. \]

\[ \int 2 \, dx = 2x + C_1(y) \]
\[ \int y \, dy = \frac{y^2}{2} + C_2(x) \]

Take \( U = 2x + \frac{y^2}{2} \).
\[ u = \langle x, y \rangle. \]

\[ \begin{align*}
\text{Ex} & \left\langle y \sin xy + 1, x \sin xy \right\rangle, \\
\int (y \sin xy + 1) \, dx &= -\cos xy + x + C \text{ only of } y, \\
\int x \sin xy \, dy &= -\cos xy + C \text{ only of } x.
\end{align*} \]

\[ u = -\cos(xy) + x \]
\[ \nabla U = \langle y \sin(xy) + 1, x \sin(xy) \rangle. \]

**Ex:** \[ \langle y \sin xy + y, x \sin xy \rangle. \]

\[ \int (y \sin xy + y) \, dx = -\cos(xy) + \frac{y^2 x}{2} + (\text{fn of } y). \]

\[ \int x \sin xy \, dy = -\cos(xy) + (\text{fn of } x). \]

\[ U = \text{can't be found} \] inconsistent
Main Theme of Last Part of course: Fundamental Theorem of Calculus.

Need to be able to integrate over more sophisticated things.

Next stop: line integrals (§ 14.2).
How is a path described? \( \vec{r}(t) = (x(t), y(t), z(t)) \).

Parametrization.

For line integrals, want \( \vec{r} \) to be piecewise \( C^1 \).

If domain of \( \vec{r} \) is \([a,b]\), can break \([a,b]\) into pieces \([a,c_0]\), \([c_0,c_1]\), \ldots, \([c_n,b]\)

\[ a \quad c_0 \quad c_1 \quad c_2 \ldots \quad b \]

so that \( \vec{r} \) is \( C^1 \) on each interval \([a,c_0]\), \([c_0,c_1]\), \ldots
i.e. $\vec{r}'(t)$ exists and is continuous.

$$\exists \vec{r}'(t) = \langle \cos t, \sin t \rangle, \quad t \in [0, 2\pi].$$

$$\vec{r}'(t) = \langle -\sin t, \cos t \rangle.$$

Continuous always!

$$\exists \vec{r}(t) = \begin{cases} 
\langle t, 0 \rangle & 0 \leq t \leq 1 \\
\langle 1, t-1 \rangle & 1 \leq t \leq 2 \\
\langle 3-t, 1 \rangle & 2 \leq t \leq 3 \\
\langle 0, 4-t \rangle & 3 \leq t \leq 4.
\end{cases}$$
Non-example

\[ P(t) = \begin{cases} 
  \langle t, 0 \rangle & \text{if } t < 0 \\
  \langle t, 1 \rangle & \text{if } t \geq 0
\end{cases} \]
The line integral of function \( f(x, y, z) \) over a curve \( C \) parametrized by \( r(t) \) with respect to arc length is:

\[
\int_C f(x, y, z) \, ds = \int_a^b f(r(t)) \| r'(t) \| \, dt.
\]