1. Midterm on Web. See web page for room assignments (same !), reminder of rules (same !).
2. Some review problems (for basic skills) are now posted on the web page.
3. Office hour today **cancelled**.
4. Midterm will be on Ch. 13. Last material will be covered today I expect.

First Additional example from cylindrical coords.
Ex. Compute

\[ \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{x^2+y^2}^{2-x^2-y^2} (x^2+y^2)^{3/2} \, dz \, dy \, dx \]

\[ x^2+y^2 \leq z \leq 2-x^2-y^2 \]

where is \( x^2+y^2 = 2-x^2-y^2 \) ?

\[ 2x^2+2y^2 = 2 \] or \[ x^2+y^2 = 1 \].

So, two paraboloids intersect when \( x^2+y^2 = 1 \) and then \( z = 1 \).
$$y^2 = 1 - x^2 \implies y = \pm \sqrt{1-x^2}$$

$y = \sqrt{1-x^2}$ is top of circle, $y = -\sqrt{1-x^2}$ is bottom of circle so disk contained in circle is

$$\mathbb{D}(x,y) \mid -1 \leq x \leq 1 \quad -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}$$

In terms of $r$, $\theta$, $z$?

$$0 \leq r \leq 1$$

$$0 \leq \theta \leq 2\pi$$

$$x^2 + y^2 \leq z \leq 2 - x^2 - y^2$$

But

$$x^2 + y^2 = (r\cos \theta)^2 + (r\sin \theta)^2 = r^2$$
\[ \int_0^{2\pi} \int_0^1 \int_0^{2-\rho^2} \rho^2 \sqrt{1-\rho^2} \, d\rho \, r \, dz \, dr \, d\theta \]

\[ = \int_0^{2\pi} \int_0^1 \int_0^{2-\rho^2} \rho^4 \, d\rho \, dz \, dr \, d\theta \]

\[ = \int_0^{2\pi} \int_0^1 \left( 2\rho^4 - 2\rho^6 \right) \, d\rho \, dz \, dr \, d\theta \]
\[ \int_{0}^{2\pi} \left( \frac{2\sqrt{2}}{5} - \frac{2\sqrt{2}}{7} \right) \, d\theta \]

\[ = \int_{0}^{2\pi} \left( \frac{2}{5} - \frac{2}{7} \right) \, d\theta \]

\[ = 2\pi \left( \frac{2}{5} - \frac{2}{7} \right) = \frac{8\pi}{35} \]
Spherical coordinates

\[ z = \sqrt{x^2 + y^2} \]

Example: ice cream cone

How to integrate over such a region?

Symmetry of rotation around the z-axis still makes useful!
Another useful measurement is angle:

\[ \phi \]

also distance from origin:

spherical coords: \((r, \phi, \theta)\).
How to express $x, y, z$ in terms of $\rho, \phi, \theta$?

Recall

\[ \hat{\rho} = \langle 0, 0 \rangle \]

\[ \hat{\rho} = \langle \cos \theta, \sin \theta \rangle \]

\[ \hat{z} = \langle 1, 0 \rangle \]

\[ \hat{z} = \cos \theta \hat{i} + \sin \theta \hat{j} \]
Now in 3D:

\[
\begin{align*}
\sin(\phi) \begin{pmatrix} \cos \theta, \sin \theta, 0 \end{pmatrix} + \\
\cos(\phi) \begin{pmatrix} 0, 0, 1 \end{pmatrix}
\end{align*}
\]

\[
= \begin{pmatrix} \sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi \end{pmatrix}
\]
To rescale green vector to get one with length $p$:

$$< p \sin \phi \cos \theta, p \sin \phi \sin \theta, p \cos \phi >$$

$$= l$$

$$< x, y, z >$$

spherical coords.

**memorize!!**

What was our ice cream cone in spherical coords?
\[ z^2 = x^2 + y^2. \]

\[ 0 \leq \rho \leq 1. \]
\[ 0 \leq \phi \leq \frac{\pi}{4} \]
\[ 0 \leq \theta \leq 2\pi \]

*unit ball*

\[ x^2 + y^2 + z^2 \leq 1. \]

\[ 0 \leq \rho \leq 1 \]
\[ 0 \leq \theta \leq 2\pi \]
\[ 0 \leq \phi \leq \pi \]
Scale factor for integrals?

Get...

\[ dx \, dy \, dz = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \]

Jacobian det. of our transformation

Ex compute
\[
\iiint_{B} \cos \left[ \frac{(x^2 + y^2 + z^2)^{3/2}}{2} \right] \, dV,
\]

B = \{ (x, y, z) \mid x^2 + y^2 + z^2 \leq 1 \}.

**Solution**

B = \{ (\rho, \phi, \theta) \mid 0 \leq \rho \leq 1, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \pi \}

Integral becomes

\[
\int_{0}^{2\pi} \int_{0}^{\pi} \int_{0}^{1} \cos \left[ \left( \rho^2 \right)^{3/2} \right] \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta
\]
\[ \int_0^{2\pi} \int_0^\pi \int_0^{\frac{1}{3} \sin(p^3)} \frac{1}{r} \sin\phi \, d\phi \, d\rho \, d\theta \]

\[ = \ldots \]