Triple Integrals (13.5 & 13.6).

"double integrals in 3D": 

\[ R = \left\{ (x, y, z) \mid a \leq x \leq b, \quad c \leq y \leq d, \quad c \leq z \leq f \right\}. \]

Define \[ \iiint_{R} f(x, y, z) \, dV \] as before.

"as before."

Computing volume of 4D inv't region under the graph.
Similarly, we can define it, meaning \[ \iiint_{R} f(x, y, z) \, dV \]
for any "reasonable" region \( R \).

In particular, we get \[ \iiint_{R} 1 \, dV = \text{vol}(R) \] for the volume.
For rectangular prisms, and even more generally, 

**Fubini's Theorem**

\[
\iiint_R f(x, y, z) \, dV = \int_a^b \int_c^d \int_e^f f(x, y, z) \, dx \, dy \, dz
\]

where

\[R = \{(x, y, z) \mid a \leq x \leq b, \quad c \leq y \leq d, \quad e \leq z \leq f\}\]

[If \( f \) is reasonable, e.g. "continuous" is enough.]
A solid bounded by 
\[ z = x^2, \quad z = 1, \quad y = 0, \quad y = 2. \]
What is its volume?

Parabollic bowl trough.

\[
\iiint_R dV = \int_0^2 \int_{-1}^1 \int_{x^2}^1 dz \, dx \, dy
\]

i.e.,
\[
R = \left\{ (x, y, z) \mid -1 \leq x \leq 1, \quad 0 \leq y \leq 2, \quad x^2 \leq z \leq 1 \right\}
\]
Integration is
\[ 
\int_{-1}^{1} \int_{0}^{2} x^2 \, dx \, dy
\]
\[ = \int_{0}^{2} \int_{-1}^{1} (1 - x^2) \, dx \, dy
\]
\[ = \int_{0}^{2} \left[ x - \frac{x^3}{3} \right]_{-1}^{1} \, dy
\]
\[ = \int_{0}^{2} \left[ (1 - \frac{1}{3}) - (-1 + \frac{1}{3}) \right] \, dy
\]
\[ = \int_{0}^{2} \left[ \frac{2}{3} + \frac{2}{3} \right] \, dy = 2 \cdot \frac{4}{3} = \frac{8}{3}
\]
Could also have done it as a double integral,

\[
\int_{-1}^{1} \int_{-1}^{1} (1 - x^2) \, dx \, dy
\]
Ex: What is $\text{vol}(R)$, where $R$ is the region between $z = 4 - x^2 - y^2$ and the $xy$ plane?

Where do the paraboloid and plane intersect?

$$0 = z = 4 - x^2 - y^2$$

i.e. $x^2 + y^2 = 4$
Let's compute vol w/ a double integral:

\[ \iint_B (4 - x^2 - y^2) \, dA \]

\[ \text{po}(\text{ar coords}) \]

\[ x = r\cos \theta , \ y = r\sin \theta \]

\[ = \int_0^{2\pi} \int_0^2 \left( 4 - (r\cos \theta)^2 - (r\sin \theta)^2 \right) r \, dr \, d\theta \]
\[ \int_{\theta=0}^{2\pi} \int_{r=0}^{2} \int_{z=0}^{\frac{1}{r^2(\cos^2 \theta + \sin^2 \theta)}} 1 \, r \, dr \, d\theta \, dz \\
= \int_{\theta=0}^{2\pi} \int_{r=0}^{2} [4 - r^2] \, dr \, d\theta \\
= \int_{\theta=0}^{2\pi} \left[ \frac{8r^4}{4} \right]_{0}^{2} \, d\theta \\
= \int_{\theta=0}^{2\pi} [8 - 4] \, d\theta = 8\pi \\

Could rewrite:

\[ R = \{ (r, \theta, z) \mid 0 \leq r \leq 2, 0 \leq \theta \leq 2\pi, 0 \leq z \leq 4 - r^2 \} \]
\[ \int_0^{2\pi} \int_0^2 \int_0^{r-1} r \, dz \, dr \, d\theta \]

I'll get some result!

How to express systematically?

"Cylindrical coordinates" transformation

\[ T(r, \theta, z) = \langle x(r, \theta, z), y(r, \theta, z), z(r, \theta, z) \rangle \]

\[ T(r, \theta, z) = \langle r \cos \theta, r \sin \theta, z \rangle \]
takes box to cylinder.
What happens to volume?

\[ \frac{\partial \mathbf{T}}{\partial r} = \langle \cos \theta, \sin \theta, 0 \rangle \]

\[ \frac{\partial \mathbf{T}}{\partial \theta} = \langle -r \sin \theta, r \cos \theta, 0 \rangle \]

\[ \frac{\partial \mathbf{T}}{\partial \varphi} = \langle 0, 0, 1 \rangle \]

Areas differ by

Volume of parallelepiped
\[ \frac{\sqrt{3}}{2} \leq x, \quad \frac{\pi}{6}, \quad \frac{\pi}{4}, \quad \frac{\pi}{3} \]

\[ \cos \theta, \quad \sin \theta, \quad \csc \theta, \quad \sec \theta, \quad \cot \theta \]

\[ \frac{1}{2}, \quad 0, \quad 0, \quad \sqrt{3}, \quad 1 \]

\[ \frac{1}{2}, \quad 0, \quad 0, \quad \sqrt{3}, \quad 1 \]
\[ \begin{align*}
\dot 0 & = \frac{r \cos^2 \theta + r \sin^2 \theta}{r (\cos^2 \theta + \sin^2 \theta)} \\
& = 1
\end{align*} \]

Therefore, the scale factor is 1.

So,
\[ dxdydz = r dr d\theta dz, \text{ i.e.} \]
\[ \iiint_{C} f(x, y, z) \, dV \]

\[ = \iiint_{D} f(r \cos \theta, r \sin \theta, z) \, r \, dr \, d\theta \, dz \]
General Transformation

\[ T(u, v, w) = \langle xu(u, v, w), y(u, v, w), z(u, v, w) \rangle \]

\[ J(T) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix} \]

Then if rectangle \( D \) in \( u, v, w \) space goes to region \( R \) in \( x, y, z \) space, we have
\[ \int_\mathbb{R} f(x, y, z) \, dV \]

\[ = \int_D \left( f(x(u, v, w), y(u, v, w), z(u, v, w)) \right) \, d(u, v, w) \]