Math 241 §BL1

Problem Set 27

(1) Suppose that $f(x)$ and $g(y)$ are continuous functions. Show that $\vec{F} = \langle f(x), g(y) \rangle$ is conservative.

(2) Show that
\[
\vec{F}(x, y) = \left\langle \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right\rangle
\]
is not conservative on the entire plane (Hint: see Problem Set 12).

(3) Consider the vector field $\vec{F}(x, y) = \langle y, x \rangle$.

(a) Show that $\vec{F}$ is conservative by finding a potential function $f(x, y)$.

(b) Show that for any real $c$, the curve defined by $f(x, y) = c$ intersects flow lines of $\vec{F}$ at right angles (it’s not necessary to find the flow lines explicitly in order to do this).

(c) Let $g(s) = c_1 e^s + c_2 e^{-s}$, where $c_1$ and $c_2$ are constants. Show that $g''(s) = g(s)$. It is a fact that every solution of the latter differential equation is of the same form as $g(s)$.

(d) Find a general formula for the flow lines of $\vec{F}$.

(e) Suppose a flow line of $\vec{F}$ passes through $(1, 1)$ at a certain time $s_0$. Conclude as in (b) that the level curve of $f$ passing through $(1, 1)$ is perpendicular to the flow line at that point using the formula you found in (d).

(4) Consider the vector field
\[
\vec{F}(x, y) = \vec{i} + x\vec{j}.
\]
on the plane. A particle is dropped onto the plane at time $t = 0$ and at position $(-2, 2)$.

(a) Describe the motion of the particle.
(b) How far is the particle from its initial position at time \( t = 4 \)?

(c) Estimate the distance that the particle has traveled at time \( t = 4 \). Hint:

\[
\frac{d}{du} \left( \frac{u}{2} \sqrt{1 + u^2} + \frac{1}{2} \ln(u + \sqrt{u^2 + 1}) \right) = \sqrt{1 + u^2}
\]