\[ \text{PS 22: Selected Answers/Partial Solns.} \]

1. \[ \text{Tab} \]

2. Assuming constant density, symmetry about \( x = 0 \Rightarrow \boxed{x = 0} \). Now,
\[
m = \rho \int \int_R 1 \, dA = \rho \left( \frac{1}{2} \right) 6 \pi = 3 \pi \rho
\]
by Problem 1.

\[
\bar{y} = \frac{1}{m} M_x = \left( \frac{1}{3\pi \rho} \right) \rho \int \int_R y \, dA
\]
(switch to elliptical coordinates) = \[
\frac{1}{3\pi} \int_0^{\pi} \int_0^1 2r^2 \sin \theta \, dr \, d\theta
\]
\[
= \left( \frac{1}{3\pi} \right) \left( \frac{4}{3} \right) = \frac{4}{9\pi}
\]
So, the center of mass is \((0, \frac{4}{9\pi})\).
(3) Let \( u = \frac{1}{x} \) and \( v = xy \). Then,

\[
UV = y^2 \quad \text{and} \quad \frac{v}{u} = x^2.
\]

So,

\[
\frac{\partial (x,y)}{\partial (u,v)} = \begin{vmatrix}
\frac{v}{u} & \frac{1}{uv} \\
\frac{v}{uv} & \frac{u}{uv}
\end{vmatrix} = \frac{-vu^{-1}}{2u} = -\frac{1}{2u}
\]

Note that \( |\frac{\partial (x,y)}{\partial (u,v)}| = \frac{1}{2u} \) because \( 1 \leq u \leq 2 \).

So, the area we want is

\[
\frac{1}{2} \iint_{1}^{2} \frac{1}{u} \, du \, dv = \frac{1}{2} \ln(2).
\]

(4) Let \( u = x-y \) and \( v = x+y \). So,

\( 0 \leq v \leq 1 \) and \( -v \leq u \leq v \). Also,

\[
\frac{\partial (x,y)}{\partial (u,v)} = \frac{1}{2}.
\]

Then,

\[
\iint_{\mathbb{R}} \exp\left(\frac{x-y}{x+y}\right) \, dA = \frac{1}{2} \iint_{-v}^{v} \exp\left(\frac{u}{v}\right) \, du \, dv
\]

\[
= \frac{e^2 - 1}{4e}.
\]
Let \( u = xy \) and \( v = x^2 - y^2 \). Then,
\[
\frac{\partial(u,v)}{\partial(x,y)} = -2(x^2 + y^2).
\]

Note that \((x^2+y^2)^2 = 4u^2 + v^2\), so that
\[
\star = -2\sqrt{4u^2 + v^2}.
\]

By the result of Problem 5,
\[
\left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \frac{1}{2\sqrt{4u^2 + v^2}}.
\]

Hence,
\[
\iint_R x^2 + y^2 \, dA = \int_1^3 \int_1^3 \frac{\sqrt{4u^2 + v^2}}{2\sqrt{4u^2 + v^2}} \, du \, dv
\]
\[
= 3.
\]