Problem. Let \( R = k[z^2, z^3] \), a subring of \( S = k[z] \), \( k \) a field. Let \( R \rightarrow S \) denote the inclusion homomorphism. Let

\[
i^{-1} : \{ \text{prime ideals} \} \rightarrow \{ \text{prime ideals} \}
\]

be defined by

\[
i^{-1}(P) = \{ r \in R \mid i(r) \in P \}\.
\]

Prove that

(i) If \( M \leq S \) is a maximal ideal, then \( i^{-1}(M) \) is a maximal ideal of \( R \).

(ii) \( i^{-1} \) induces a bijection

\[
\{ \text{maximal ideals} \} \rightarrow \{ \text{maximal ideals} \}
\]

\[
\{ M \in R \} \rightarrow \{ N \in S \}.
\]
We begin with:

**Lemma** Suppose $JS$ is a proper ideal.

Then the ideal $JS \subseteq S$ generated by $J$ is a proper ideal of $S$.

Proof. Since

$$k[x] = \{ \frac{a}{g} + az \mid f \in k[x^2, x^3], a \in k \},$$

one can easily check that

$$JS = J + zJ = \{ j_1 + zj_2 \mid j_1, j_2 \in J \}.$$

Suppose $JS$ is not proper, i.e.

$$JS = S.$$ Then $1 \in JS$, so

$$1 = j_1 + zj_2$$

for some $j_1, j_2 \in J$.

Then

$$z = z \cdot 1 = zj_1 + z^2j_2,$$ but

$$zj_1, z^2j_2 \in J,$$ so $z^2 \in J$. 


Which ideals of \( R \) can contain \( z^2 \)?

Well, \((z^2) = \{ t z^2 \mid t \in R \}\) is the span of \( z^2, z \cdot z = z^3, z \cdot z^2 = z^4, \ldots \) So,

\[
\begin{align*}
R/(z^2) &= \left\{ a + bz^3 + (z^2) \mid a, b \in k \right\}.
\end{align*}
\]

Now \((a + bz^3 + (z^2))(a - bz^3 + (z^2)) = a^2 + (z^2)\),

so if \( a \neq 0 \) then \( a + bz^3 + (z^2) \) is a unit in \( R/(z^2) \). Thus, a proper ideal of \( R/(z^2) \), which corresponds to a proper ideal of \( R \) containing \( z^2 \), could only be \( (z^2) \) or \( \left\{ bz^3 + (z^2) \mid b \in k \right\} = (z^2, z^3) \).

In either case \( J = (z^2) \) or \( J = (z^2, z^3) \), it's easy to see that \( J \cap S \neq S \), a contradiction. So \( J \) is always proper. \( \square \)
Now, we return to solving our problem. It follows from the lemma that every proper ideal of \( R \) is contained in a proper ideal of \( S \). Given a maximal ideal \( N \subseteq R \), let \( M \subseteq S \) be any maximal ideal of \( S \) with \( N \subseteq M \). Then

\[ N \subseteq MRR, \text{ but } M/R = i^{-1}(M), \]

and \( i^{-1}(M) \) is a proper ideal of \( R \). So, by definition of maximal ideal, we must have \( N = i^{-1}(M) \).

It remains to show

1. if \( M \) is a maximal ideal, \( i^{-1}(M) \) is maximal.
2. if \( i^{-1}(M_1) = i^{-1}(M_2) \) for maximal ideals \( M_1, M_2 \) then \( M_1 = M_2 \).
I'm going to do the next bit in greater generality than is really necessary for this problem.

Lemma. Suppose a commutative ring w/1
S is a PID, and f₁, f₂ ∈ S are irreducible. Suppose there is no unit u ∈ S such that f₁ = uf₂. Then

\[(f₁) \cap (f₂) = (f₁f₂)\].

Proof. f₁f₂ ∈ (f₁) ∩ (f₂), so

\[(f₁f₂) ⊆ (f₁) ∩ (f₂)\]. Conversely,

if h ∈ (f₁) ∩ (f₂), say

f₁g₁ = h = f₂g₂ where g₁, g₂ ∈ S

use the fact that S is a UFD to write
\[ g_1 = g_{1,1} \ldots g_{1,k-1} \text{ and} \]
\[ g_2 = g_{2,1} \ldots g_{2,k-1} \]

with each \( g_{1,j} \) and \( g_{2,j} \) irreducible.

Then
\[ f_1 g_{1,1} \ldots g_{1,k-1} = h = f_2 g_{2,1} \ldots g_{2,k-1} \]

and by our hypothesis and the definition of \( \text{UPD} \), \( f_1 = u g_{2,l} \) for some \( l \) and some unit \( u \in S \). Then
\[ h = f_1 f_2 g_{2,1} \ldots g_{2,k-1} u^{-1} g_{2,k+1} \ldots g_{2,k+t} \]

so \( f_1 f_2 \) divides \( h \), i.e.
\[ h \in (f_1 f_2). \]
Returning to our problem, suppose $M_1 = (f_1), M_2 = (f_2)$ are maximal ideals in $S = k[x,y]$. So, $f_1$ and $f_2$ are irreducible. If $M_1 \neq M_2$, then the lemma gives $M_1 \cap M_2 = (f_1 f_2)$.

Suppose $i^{-1}(M_1) = J = i^{-1}(M_2)$.

Then $J \subseteq M_1 \cap M_2$. Now, $J = i^{-1}(M_1) = (f_1) \cap R$, so $z f_1 \subseteq J$. But then

$$z^2 f_1 = h f_1 f_2$$
for some $h \in S$,

so $z^2 = h f_2 \Rightarrow f_2 = z$. Symmetrically, we also get $f_1 = z$, a contradiction. So if $M_1, M_2$ are maximal ideals of $S$, $i^{-1}(M_1) \neq i^{-1}(M_2)$. 


Finally, it remains to show that if $M/\mathfrak{S}$ is maximal then $\mathfrak{c}^{-1}(M)$ is maximal. Suppose $M = \langle f \rangle$ where $\mathfrak{f}(\mathfrak{z}) \in \mathfrak{S} = k[\mathfrak{z}]$ is irreducible. Then $\mathfrak{S}/\mathfrak{M} = k[\mathfrak{z}] / \langle f \rangle$ is a field. The kernel of the homomorphism

$$R \to \mathfrak{S}/\mathfrak{M}$$

$$\alpha(g) = g + \langle f \rangle$$

is exactly

$R \cap \langle f \rangle = \mathfrak{c}^{-1}(M)$. So

$R / \mathfrak{c}^{-1}(M)$ is a subring of $\mathfrak{S}/\mathfrak{M}$.

We want to show that $R / \mathfrak{c}^{-1}(M)$ is a field, then we can conclude that $\mathfrak{c}^{-1}(M)$ is a maximal ideal.
We have 

\[ k \subseteq R/i^{-1}(m) \subseteq S/m = k[z]/(f). \]

We'll prove:

**Lemma** Suppose we have a comm ring \( R \) and injective homomorphisms

\[ k \subseteq T \subseteq k[z]/(f) \]

such that the composite map

\[ k \rightarrow k[z]/(f) \rightarrow C \]

is the usual map

\[ C \rightarrow C + (f). \]

Suppose \( k[z]/(f) \) is a field. Then \( T \) is also a field.

**Proof.** Let \( x \in T \), \( x \neq 0 \). We'll show \( x \) is a unit in \( T \) (i.e., its inverse lies in \( T \)).

Write \( f = a_0 + a_1 z + \cdots + a_n z^n \), \( a_n \neq 0 \).

Then \( 1, z, z^2, \ldots, z^{n-1} \) span \( S/m \) as a
$k$-vector space, and $S/M$ has dimension $n$. Thus, $1, x, x^2, \ldots, x^n$ forms a linearly dependent set, say

$$(*) \quad c_k x^k + \ldots + c_n x^n = 0 \text{ with } c_k \neq 0,$$

and $c_k, \ldots, c_n \in k$.

Then, since $T$ is a domain (it is a subring of a field), we have

$$c_k x^k + c_{k+1} x^{k+1} + \ldots + c_n x^n = 0 \, (\text{we cancelled } x^k \text{ from both sides of } (*) \,).$$

So $c_k = - (c_{k+1} x^{k-1} + \ldots + c_n x^{n-k})$ and

$$x \cdot \left[ (-c_{k+1} - \ldots - c_n x^{n-k-1}) \cdot (c_k^{-1}) \right] = 1.$$

This lies in $T$. \qed
This proves that $R/\overline{\gamma}(m)$ is a field, so $\overline{\gamma}(m)$ is a maximal ideal as desired.