Mathematics 427, Homework 4 problem

Fix a prime number $p$. Let $(\mathbb{Z}/p\mathbb{Z})^\times$ denote the set of nonzero elements of $\mathbb{Z}/p\mathbb{Z}$.

Give the Cartesian product set

$$G = (\mathbb{Z}/p\mathbb{Z})^\times \times (\mathbb{Z}/p\mathbb{Z})$$

a product operation by letting

$$(a, b) \cdot (c, d) = (ac, ad + b).$$

(1) prove that $G$ is a group with this operation.
(2) What is the order of $G$?
(3) Prove that the subset

$$K = \{(1, b) \mid b \in \mathbb{Z}/p\mathbb{Z}\}$$

is a normal subgroup of $G$.
(4) Consider the case $p = 5$. Does $G$ have any nontrivial homomorphisms onto a group of order 5?

Remark. $G$ is called the group of affine transformations of the line $A^1_{\mathbb{F}_p}$. 