Let $\mathcal{M} \rightarrow \mathbb{P}^n$ be a line bundle over $\mathbb{P}^n$, with a choice of trivializations over the open sets $U_i \subset \mathbb{P}^n$ by bijections $\tilde{\phi}_i = (\phi_i, \Phi_i) : \pi^{-1}(U_i) \rightarrow \mathbb{A}^n \times k$.

We can then form the transition functions $\psi_{ji} = \Phi_j \circ \Phi_i^{-1} : U_i \cap U_j \rightarrow \text{GL}_1(k)$.

(Recall that, for each point $x \in U_i \cap U_j$, the map $\Phi_i(x)$ is a linear isomorphism from the fiber $\pi^{-1}(x)$ to $k$, so the composite $\Phi_j(x) \circ \Phi_i^{-1}$ is a linear isomorphism from $k$ to $k$, hence is given by scalar multiplication by a nonzero element of $k$.)

Prove that a collection of functions $F_i \in k[u], i = 0, \ldots, n$ as in the lectures, determine a regular section of the line bundle $\mathcal{M}$ by the procedure described in the lectures if and only if $F_j \circ \phi_j = (\psi_{ji}) \cdot (F_i \circ \phi_i)$ on $U_i \cap U_j$ for all $i, j$. Here $\phi_i : U_i \rightarrow \mathbb{A}^n$ is the usual coordinate chart on the open set $U_i$ of $\mathbb{P}^n$.

Let $\mathcal{L} \rightarrow \mathbb{P}^n$ be a line bundle, then the transition functions for the dual line bundle $\mathcal{L}^* = \{(x, v) \mid x \in \mathbb{P}^n, v \in \text{Hom}_k(\pi^{-1}(x), k)\}$ are given by $\psi_{ji}^* = 1/\psi_{ji}$, where $\psi_{ji}$ are the transition functions for $\mathcal{L}$.

Prove that if $\mathcal{L} \rightarrow \mathbb{P}^n$ is a line bundle, with transition functions $\psi_{ji}$, then the transition functions for the line bundle $\mathcal{L} \otimes \mathcal{O}^m = \{(x, v) \mid x \in \mathbb{P}^n, v \in (\pi^{-1}(x)) \otimes \mathcal{O}^m\}$ are given by $\psi_{ji}^m$.

Prove that in the scheme $\mathbb{P}^1_k = \text{Proj}(k[x_0, x_1])$ with $k$ a field, the homogeneous ideals $(x_0)$ and $(x_0^2, x_0 x_1)$ define the same (one-point, reduced) closed subscheme. [Hint: find an open set of the form $\mathcal{D}(f)$ that contains this closed subscheme, then calculate the two relevant localizations.]

Suppose that $f \in k[x_0, x_1]$ is a (not necessarily irreducible) nonzero homogeneous polynomial of degree $d$. What is the dimension of the degree $N$ component of $k[x_0, x_1]/(f)$ for $N$ large? How could you interpret this in terms of the corresponding closed subscheme of $\mathbb{P}^1_k$? Describe the closed subscheme in the case $f = x_0^3$.

Do problems 2.10, 2.11, and 2.12 from Chapter I, section 2 of Hartshorne.