Divisors and Line Bundles

A prime (Weil) divisor on a scheme $X$ is a closed integral subscheme of codimension 1.

A Weil divisor on $X$ is an element of the free abelian group on the prime Weil divisors.

Note: We usually require $X$ to be regular in codimension 1, i.e., for every $x \in X$ such that $\mathcal{O}_x, x$ is 1-dimensional, $\mathcal{O}_x, x$ is a regular ring. Such a ring is then a DVR!

We'll always now assume $X$ is integral, separated scheme of finite type over a field $K$.

A divisor $\sum n_i [D_i]$ is effective if each $n_i \geq 0$. 
Sources of Divisors

Y \subseteq X a prime divisor, \ y \in Y the generic point of \ Y. Then \ \mathcal{O}_{X,y} is a regular local 1-dimensional ring, i.e. a DVR.

Proof. Consider an affine open set

\[ U = \text{Spec}(R) \subseteq X \] such that \ Y \cap U \neq \emptyset.

Then \ Y \cap U = \text{Spec}(R/I) for some ideal \ I \subseteq R; since \ Y is integral, \ I is a prime ideal of \ R. The generic point of \ Y \cap U, and hence at \ y, is

\[ (0) \in \text{Spec}(R/I); \text{ the image of this point under the closed immersion } \text{Spec}(R/I) \hookrightarrow \text{Spec}(R) \]

is the prime \ I \subseteq \text{Spec}(R). Hence

\[ \mathcal{O}_{X,y} = R_\mathcal{I}. \text{ One checks from the definition of dimension that} \]

\[ 1 = \text{codim}(Y, X) = ht(I), \text{ so} \]
the localization \( R_I \) is 1-dimensional. It is regular by our blanket assumption on \( X \). \( \square \).

Let \( \nu : K(\mathcal{O}_X, m) \to \mathbb{Z} \) be the corresponding surjective valuation.

**Note:** In the notation of the above proof, \( K(\mathcal{O}_X, m) = K(R_I) = K(R) \).

This is equal to the function field \( K(X) \) of \( X \):

\[
K(X) = \lim_{\text{open} \atop \emptyset \neq U \subseteq X} \mathcal{O}_X(U).
\]

**Exercise:** If \( \mathbf{1} \) is the generic point of \( X \),

\[
K(\mathbf{1}) = \mathcal{O}_{X, \mathbf{1}}.
\]
Lemma \( \text{let } f \in K(X)^* \); then \( v_f(f) = 0 \) for all but finitely many prime divisors \( v \).

Proof. Let \( U = \text{Spec}(R) \subseteq X \) be affine open. Then \( X \setminus U \) contains at most finitely many prime divisors; since \( X \) is separated over \( k \) by assumption, if \( V \subseteq X \) is any affine open subset then \( V \setminus U \) is affine open in \( V \). Cover \( X \) by finitely many affine open subsets \( V_i \) (recall \( X \) is of finite type over \( k \)).

If \( X \subseteq X \) is a prime divisor then \( Y \setminus V_i \) is a nonempty prime divisor of \( V_i \) for some \( i \). So it suffices to check that \( V_i \setminus (Y \setminus U) \) contains only finitely many prime divisors for each \( i \). Write \( V_i = \text{Spec}(R_i) \). Then, shrinking \( V_i \setminus U \) if necessary, we may assume that
\[ V_i \cap U = D(f_i) = \text{Spec} \left( R_i \left[ \frac{1}{f_i} \right] \right) \subseteq \text{Spec} \ R_i. \]

Then any prime divisor of \( \text{Spec} \left( R_i \right) \) contained in \( V_i \setminus (V_i \cap U) \) contains \( f_i \). There can be only finitely many primes of \( R_i \) that are minimal among nonzero primes and contain \( f_i \); see Exercise 5.1.2 of Eisenbud. This completes the proof. \( \square \)

**Def.** Given \( f \in K(X)^x \), the *divisor* of \( f \)

\[ (f) = \sum_{Y \subseteq X} v_Y(f) Y, \]

where \( Y \) ranges over all prime divisors of \( X \).

A divisor of the form \( (f) \) for some \( f \in K(X)^x \) is called *principal*.

**Def.** Two divisors \( D, D' \) are *linearly equivalent* if \( D - D' \) is principal.
Note. Given \( f, g \in K(X)^x \), we have
\[
(fg) = (f) + (g),
\]
\[
(g^{-1}) = -(g).
\]

So, \( f \mapsto (f) \) defines a group homomorphism
\[
(K(X)^x, \cdot) \longrightarrow (\text{Div}(X), +)
\]
where \( \text{Div}(X) \) is the group of divisors on \( X \).

**Def.** \( \text{Cl}(X) = \text{Div}(X)/\langle \text{principal divisors} \rangle \).
Example \( X = \mathbb{P}_k^n, \ k \) a field.

Let \( D = \sum n_i y_i \), \( y_i \) a reduced, irreducible hypersurface in \( \mathbb{P}_k^n \) of degree \( d_i \).

Lemma \( Y \subseteq \mathbb{P}_k^n \) a reduced and irreducible hypersurface. Then

\( Y = V(f) \) for a unique, up to multiplication by an element of \( k^* \), irreducible homogeneous \( f \in k[x_0, \ldots, x_n] \).

Proof.
So, back to $\mathbb{P}^n$, take

$$D = \sum n_i \cdot \mathcal{O}_{\mathbb{P}^n}(f_i) = \mathcal{O}_{\mathbb{P}^n}(\sum n_i \cdot V(f_i))$$

with $f_i \in K[x_0, \ldots, x_n]_{d_i}$.

**Def** \(\deg(D) = \sum n_i d_i\).

This gives a homomorphism

$$\deg : \text{Div}(\mathbb{P}^n) \to \mathbb{Z}.$$

**Prop**

1) \(\deg(f) = 0 \quad \forall f \in K(\mathbb{P}^n)^*\).

2) Let $h \in K[x_0, \ldots, x_n] \setminus \{0\}$,

$$H_h = V(h)$$

the corresponding hyperplane.

Then $D = \sum n_i \cdot \mathcal{O}_{\mathbb{P}^n}(f_i)$ is linearly equivalent to $\deg(D) \cdot H_h$.

3) \(\text{Cl}(\mathbb{P}^n) \cong \mathbb{Z}\).
We first define the divisor associated to a rational section of a line bundle.

Let $U \subseteq X$ be a nonempty open set, $L$ an invertible sheaf on $X$, $s \in H^0(U, L)$ a nonzero section.

Suppose $Y \subseteq X$ is a prime divisor. Let $\eta \in Y$ be the generic point of $Y$. Choose an open neighborhood $V$ of $\eta$ in $X$, sufficiently small that $L|_V \cong \mathcal{O}_V$, and choose an isomorphism $\varphi : L|_V \cong \mathcal{O}_V$.

Define $v_Y(s) = v_Y(\varphi(s))$.

Note that $\varphi(s) \in \mathcal{O}_X(U \cap V) \subseteq \mathbb{K}(X)^*$.  

Lemma $v_Y(s)$ does not depend on the choice of $v$ or $\varphi$.

Proof. If $W \subseteq V \subseteq X$ with $\eta \in W$, then  

$v_Y(\varphi(s)_{|_VW}) = v_Y(\varphi(s)_{|_VW})$, so $v_Y(s)$ does not depend on the choice of $V$. 

Similarly if \( \phi, \phi' \) are two isomorphisms, then 
\( \phi^{-1}(1) = f(\phi')^{-1}(1) \) for some unit 
\( f \in \mathcal{O}(V)^{\times} \). Writing \( t = \phi^{-1}(1) \), we get 
\( \phi(t) = 1 \), \( \phi'(t) = f \). Thus, if 
\( S/uvw = g \cdot t/uvw \), we get 

\[ v_\gamma(\phi(s)) = v_\gamma(g) \; , \]

\[ v_\gamma(\phi'(s)) = v_\gamma(fg) = v_\gamma(g) + v_\gamma(f) \; . \]

But if \( f \in \mathcal{O}(V)^{\times} \) then the image of \( f \) in 
\( \mathcal{O}_X/\mathfrak{m} \) is a unit \( \Rightarrow v_\gamma(f) = 0 \). \( \square \)

**Def:** The divisor of \( s \) is \( (s) = \sum_{\gamma \in \mathcal{X}} v_\gamma(s) \gamma \).

**Note:** If \( s \in H^0(U, L) \), \( s' \in H^0(U', L') \) then \( ss' \in H^0(U \cap U', L \otimes_{\mathcal{O}_X} L') \) and 
\( (ss') = (s) + (s') \).
Proof. Choose trivializations
\[ Y \colon L|_V \stackrel{\sim}{\longrightarrow} \mathcal{O}_V, \]
\[ Y' \colon L'|_V \stackrel{\sim}{\longrightarrow} \mathcal{O}_V. \]

We get a trivialization
\[ (L \otimes_{\mathcal{O}_X} L')|_V \stackrel{\otimes_Y}{\longrightarrow} \mathcal{O}_V \otimes \mathcal{O}_V = \mathcal{O}_V \]
\[ u \otimes v \mapsto uv. \]

Then
\[ v_Y (s \otimes s') \overset{\text{def}}{=} v_Y (Y(s) \otimes Y(s')) = v_Y (Y(s)Y'(s')) \]
\[ = v_Y (Y(s)) + v_Y (Y'(s')) = v_Y (s) + v_Y (s'). \] \[ \square. \]

Similarly, if \( \phi \in H^0(U, L) \), then there is an, possibly smaller, open set \( V \) and a section \( s^{-1} \in H^0(V, L^* \) such that, under the isomorphism
\[ L \otimes_{\mathcal{O}_X} L^* = \mathcal{O}_X, \]
we have
\[ s \otimes s^{-1} \mapsto 1. \]

Fact \( (s^{-1}) = -(s) \).

The proof is similar.
Lemma. If \( f \in k[x_0, \ldots, x_n] \) is irreducible, \( S_f \in H^0(\mathbb{P}^n, \mathcal{O}(d)) \) the corresponding section, then \( (S_f) = V(f) \).

Proof. Let \( Y \subseteq \mathbb{P}^n \) be a prime divisor; by an earlier lemma, \( Y = V(F) \) for some irreducible \( F \in k[x_0, \ldots, x_n] \) (homogeneous). Let \( n \in Y \) be the generic point of \( Y \), and choose \( i \) such that \( n \in D_+(x_i) \). Restricting to \( D_+(x_i) = \text{Spec } k[s_0, \ldots, s_n/x_i] \), we find that, under the usual trivialization of \( \mathcal{O}(d) \) on \( D_+(x_i) \), \( S_f \) corresponds to \( \gamma = f(s_0/x_i, \ldots, s_n/x_i) \in k[s_0/x_i, \ldots, s_n/x_i] \) which lies in \( R \). Let \( \tilde{F} = F(s_0/x_i, \ldots, s_n/x_i) \in R \), and \( \mathfrak{p} = (\tilde{F}) \) the prime it generates — note that \( \gamma \) and \( \tilde{F} \) are both irreducible in \( R \) ! Then \( Y \subseteq R \subseteq R_\mathfrak{p} \), so \( \nu_{\mathcal{O}(d)}(S_f) = \nu_{\mathcal{O}(d)}(\tilde{F}) \geq 0 \), and
\( v(V(F)) \geq k \) if \( \tilde{f} \in (\tilde{F}^k) \).

But \( \tilde{f}, \tilde{F} \) irreducible implies

\( \tilde{f} \in (\tilde{F}^k) \) iff \( k = 1 \) and \( \tilde{f} = u \tilde{F} \)

for some unit \( u \in R^* = k^* \). Thus,

\[ v(V(F)) (\tilde{c} \tilde{e}) = \begin{cases} 1 & \text{if } F = c \tilde{f} \text{ for some } c \in k^* \\ 0 & \text{otherwise} \end{cases} \]

This proves the lemma.$^\top$

We now return to \( R^* \).
Proof of Prop about \( P^n \)

(2) Write \( D = \sum n_i V(f_i) \) as above.

Then \( D - \deg(D) H_\ell = \left( \prod f_i^{n_i} \right) \frac{1}{\ell^{\deg(D)}} \)

by the previous lemmas about sections

\[ \sum f_i \in H^0 \left( \mathbb{P}^n, \mathcal{O}(\ell) \right), \quad \ell \in H^0 \left( \mathbb{P}^n, H_\ell, \mathcal{O}(\ell-1) \right). \]

(1) Let \( D_+(y) = U \subseteq \mathbb{P}^n \) be affine open, and suppose \( f \in K(\mathbb{P}^n)^* \) satisfies

\[ f \in \mathcal{O}_{\mathbb{P}^n}(U). \]

Then \( f \in k[x_0, \ldots, x_n](y) \),

i.e. \( f = y^k g \) for some \( h \in k[x_0, \ldots, x_n] \).

Write \( h = \prod h_i \) and \( g = \prod g_j \)
as products of irreducible factors.

Then

\[ (\ell) = \sum (h_i) - k \sum (g_i) \]

and

\[ \deg((\ell)) = \sum \deg(h_i) - k \sum \deg(g_i) \]

\[ = \deg(h) - k \deg(g) = 0. \]
(3) Fix $\ell = x_0$. Then

$$\text{Cl}(X) \longrightarrow \mathbb{Z}$$

$D \longmapsto \deg(D)$ is a homomorphism by (1), and $\deg(\ell x_2) = 1$ so it's surjective. Finally $\deg(D) = \deg(d \ell x_2)$.

$\Rightarrow D$ is lin. equiv. to $d \ell x_2$ so it's injective as well. □