Def: Let \( Y \) be any scheme. We let
\[
P^n_Y \overset{\text{def}}{=} \mathbb{P}^n \times_{\text{Spec} \mathbb{Z}} Y,
\]
where \( \mathbb{P}^n \) is \( \text{Proj}(\mathbb{Z}[x_0, \ldots, x_n]) \).

Exercise: If \( Y = \text{Spec} R \), then
\[
P^n_Y = \text{Proj}(R[x_0, \ldots, x_n]).
\]

A morphism of schemes \( X \to Y \) is \textit{projective} if it factors as
\[
\begin{array}{ccc}
X & \xrightarrow{i} & P^n_Y \\
\downarrow f & & \downarrow \pi \\
Y & & 
\end{array}
\]
where \( i \) is a closed immersion and \( \pi \) is the projection from \( P^n_Y = \mathbb{P}^n \times_{\text{Spec} \mathbb{Z}} Y \) to \( Y \).
Roughly, a projective morphism is one whose fibers are closed subschemes of projective space (in a reasonable way as $y \in Y$ varies).

**Exercise** If $X \to \mathbb{P}^n_Y$ is a closed immersion, then taking fibers over $y \in Y$, i.e. fiber products along

$$\text{Spec}(k(y)) \to Y,$$

gives a closed immersion $X_y \to \mathbb{P}^n_{k(y)}$.

**Example** Define a closed subscheme

$$C \subseteq \mathbb{P}^2_{k[t]} \times \mathbb{A}^1_k = \mathbb{P}^2_{k[t]} \times_k \mathbb{A}^1_{k[t]}$$

by

$$V(y^2z - x^3 - x^2z - tz^3)$$

$$y^2z - x^3 - x^2z - tz^3 \in k[t][x, y, z].$$
Each fiber $C_\lambda$ over $\lambda \in k$ is a curve,
\[ C_\lambda = V(y^2z - x^3 - x^2z - \lambda z^3). \]

Its intersection with the open set $U(\mathbb{A}) \subseteq \mathbb{P}^2_k$ is the cubic curve given by
\[ y^2 - x^3 - x^2 - \lambda. \]

The map $C \rightarrow \mathbb{A}^1$ given by the composite $C \rightarrow \mathbb{P}^2_k \times \mathbb{A}^1 \rightarrow \mathbb{A}^1$
is projective.

**Def.** The invertible sheaf $\mathcal{O}_{\mathbb{P}^n_k}(1)$ is the pullback of $\mathcal{O}_{\mathbb{P}^n_k}(1)$ along the projection $\mathbb{P}^n_k \times \mathbb{A}^1 \rightarrow \mathbb{P}^n_k$.
Def A morphism \( X \to Z \) is an immersion if it factors as
\[
X \to W \to Z \text{ where } W \to Z
\]
is a closed immersion and \( X \to W \) is an open immersion.

Given an immersion \( X \hookrightarrow \mathbb{P}^n_Y \) for some scheme \( Y \), one can pull back \( \mathcal{O}_{\mathbb{P}^n_Y}(1) \) to get an invertible sheaf \( i^* \mathcal{O}(1) \) on \( X \).

Example \( S = k[x_0, \ldots, x_n] \), \( I \subseteq S \) a homogeneous ideal, \( X = \text{Proj}(S/I) \). Then
\[
i^* \mathcal{O}(1) = \mathcal{O}_{\text{Proj}(S/I)}(1)
\]
on \( \text{Proj}(S/I) \).

Def \( f : X \to Y \) a morphism. An invertible sheaf \( \mathcal{L} \) on \( X \) is relatively (to \( f \)) very ample if there is an immersion
\[
i : X \hookrightarrow \mathbb{P}^r_Y \text{ for some } r
\]
such that \[ X \xrightarrow{i} P^k \]

commutes and \[ \Omega \cong \mathcal{O} \]

Remark. The study of line bundles / invertible sheaves on varieties or schemes helps one understand both their intrinsic and extrinsic geometry.

One example of this philosophy is if one has a projective variety \( Y \subseteq P^n_k \), one gets an invertible sheaf \( i^*(\Omega(1)) \) on \( Y \).

We have a surjection \( H^0(\mathcal{O}(1)) \rightarrow i^* i^*(\mathcal{O}(1)) \), hence a homomorphism

\[
\begin{align*}
&k[x_0, \ldots, x_n] = H^0(\mathcal{O}(1)) \rightarrow H^0(i^* i^*(\mathcal{O}(1))) \\
&= H^0(i^*(\mathcal{O}(1)))
\end{align*}
\]
What is the zero locus in \( Y \) of the section \( s_x \) of \( \mathcal{O}(1) \), associated to a non-zero \( f \in k[x_0, \ldots, x_n] \)?

Write \( Y = \text{Proj} \left( S/I \right) \), \( S = k[x_0, \ldots, x_n] \).

Over \( U_i = D(x_i) \subseteq \mathbb{P}^n_k \), we have

\[
\begin{align*}
C_{\mathbb{P}^n_k}(i)(U_i) & \longrightarrow \left( \mathbb{P}^n_k \right)(U_i) \\
| & | \\
X_i & \longmapsto \left( \mathbb{P}^n_k \right)(U_i) \\
| & | \\
X_i & \longmapsto \left( \mathbb{P}^n_k \right)(U_i)
\end{align*}
\]

Now take \( f \in S_1 \); it gives \( f \in x_i S_+(x_i) \)

as well, hence, identifying \( X_i \left( \mathbb{P}^n_k \right)(x_i) \)

as a free \( \left( \mathbb{P}^n_k \right)(x_i) \)-module via \( x_i \mapsto 1 \).

The zero locus of \( s_x \) is the subscheme.
V(\mathcal{I}_x) \leq \text{Spec} \left( (S/I)_x \right).

This closed subscheme has a more global description: it's just

$$\text{Proj} \left( (S/I) + G \right) \subseteq \text{Proj}(S) = \mathbb{P}^n_k.$$  

**Conclusion**  
The zero-locus of the section \( \mathcal{I}_f^* \mathcal{O}(1) \) is the scheme-theoretic intersection of

\( Y = \text{Proj}(S/I) \) and \( V(f) \) in \( \mathbb{P}^n_k \).

In other words, it is the intersection of \( Y \) and the hyperplane \( [f=0] \).

The collection of such hyperplane sections of \( Y \) plays an important role in the study of how \( Y \) is embedded in \( \mathbb{P}^n_k \).

Note that the immersion \( Y \subseteq \mathbb{P}^n_k \) gives both a very ample invertible sheaf \( \mathcal{I}_f^* \mathcal{O}(1) \).
on $Y$ and a space of sections

\[
\text{image } (H^0(P^n, \mathcal{O}(1)) \to H^0(Y, i^*\mathcal{O}(1)))
\]

of $i^*\mathcal{O}(1)$.

\textbf{Fact} If $Y$ is a scheme and

$Y \subseteq P^n$ is a closed immersion,

one can reconstruct $i$ from the data of

1. $i^*(\mathcal{O}(1))$

and

2. $\text{im } (H^0(P^n, \mathcal{O}(1)) \to H^0(Y, i^*\mathcal{O}(1)))$. 

One crucial fact about projective schemes (closed subschemes of projective space) is that the spaces of global sections of coherent sheaves on them are finite-dimensional.

**Theorem.** Let $k$ be a field, $A$ a finitely generated $k$-algebra, $X \to \text{Spec}(A)$ a projective morphism, $\mathcal{F}$ a coherent sheaf on $X$. Then $H^0(X, \mathcal{F})$ is a finitely generated $A$-module.

We'll focus on the proof in the case $A = k$.

Note that $H^0(X, \mathcal{F}) = H^0(\mathbb{P}^n_A, i_* \mathcal{F})$ where $i : X \to \mathbb{P}^n_A$ is a closed immersion.

Since $i_* \mathcal{F}$ is coherent, it suffices to check for $X = \mathbb{P}^1$. 

ample-sheaves Page 9
We need the following lemma.

Lemma: Given a coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$ and a section $s \in H^0(U_i, \mathcal{F}(U_i))$, there is some $m > 0$ such that $x_i^m s \in H^0(U_i, \mathcal{F}(m) |_{U_i})$
extends to a section in $H^0(\mathbb{P}^n, \mathcal{F}(m))$.

Proof. Let $\mathcal{F} = \bigoplus_{m \in \mathbb{Z}} H^0(\mathcal{F}(m))$.

Then we already saw that $\Gamma(X, \mathcal{F}) \cong \mathcal{F}$,
so $H^0(U_i, \mathcal{F}(U_i)) = \mathcal{F} (x_i)$. If $s \in \mathcal{F}(U_i)$, it is represented by some $a/x_i^m$, $a \in \mathcal{F}_m$, so $x_i^m s$ is represented by $a \in \mathcal{F}_m$, which is a global section of $\mathcal{F}(m)$. \[\square\]
Cor. There exists $m > 0$ and $k$ such that there is a surjective morphism

$$
\mathcal{O}(-m)^k \to \mathcal{F}.
$$

Proof. Find sections $s_{i,1}, \ldots, s_{i,k}$ of $\mathcal{F}$ over $U_i$ such that

$$
\mathcal{O}_{U_i}^k \to \mathcal{F}
$$

$$(k_1, \ldots, k_k) \mapsto \sum_{j} \overline{s}_{i,j}$$

is surjective. Choosing $m$ sufficiently large, all these sections give sections $x^n s_{i,j}$ that extend to elements $\overline{s}_{i,j}$ of $H^0(P^n, \mathcal{O}(m))$. By construction, the induced map

$$
\mathcal{O} \to \mathcal{F}(m)
$$

$$(k_1, \ldots, k_k) \mapsto \sum_{j} \overline{s}_{i,j}$$

is surjective over $U_i$ for each $i$. Indeed,
\[ \sum_{i,j} \overline{f}_{i,j} \overline{s}_{ij} = \sum_{i,j} f_{i,j} x_i s_{ij} = x_i \sum_{j} s_{ij} \]

Since \( x_i \) is a unit in \( \mathcal{O}(U_i) \) and the \( s_{ij} \) generate \( \mathcal{F}(U_i) \), the claim follows. Hence \( \mathcal{O}(\ell) \to \mathcal{F}(\mathbb{P}^n) \) is also surjective over \( \mathbb{P}^n \). Finally, we set \( k = \sum \ell_i \) and tensor with \( \mathcal{O}(-m) \) to get a surjective morphism
\[ \mathcal{O}(-m)^k \to \mathcal{F}. \]

Now, by an earlier-mentioned result of Serre, \[ H^0(\mathcal{O}(-m)^k, \ell) \to H^0(\mathcal{F}(\ell)) \]
\[ \downarrow \]
\[ H^0(\mathcal{O}(\ell-m)^k) \]

is surjective for \( \ell > 0 \). Hence \( M = \text{Image} \left( \Gamma_{\mathbb{P}^n}(\mathcal{O}(-m)^k) \to \Gamma_{\mathbb{P}^n}(\mathcal{F}) \right) \) is a graded submodule of \( \Gamma_{\mathbb{P}^n}(\mathcal{O}) \) that
equals $\Gamma^*_x(E)$ itself in high degree.

Note we get an exact sequence

$$0 \rightarrow M \rightarrow \Gamma^*_x(E) \rightarrow Q \rightarrow 0$$

of graded modules, with $Q_l = 0$ for all $l$ sufficiently large. Since

then, for every $q \in Q$, there is $N \gg 0$ such that $x^Nq = 0$, we get that

$$M_{x^N} \cong (\Gamma^*_x F)_{x^N} \neq 0,$$

hence

$$M_{(x^N)} \cong (\Gamma^*_x F)(x^N) \neq 0.$$  

Hence $\hat{M} = \bar{F}$.

By construction, $M$ is a finitely generated $k[x_0, \ldots, x_n]_m$-module. We may filter $M$ by graded submodules

$$0 \subset M_1 \subset M_2 \subset \cdots \subset M_s = M$$
such that \( M_j/M_{j-1} \cong K[x_0, \ldots, x_n]/\mathcal{I}_j \)

for some homogeneous prime \( \mathcal{I}_j \).

Claim. Suppose \( H^0(P^n, \widetilde{\mathcal{M}}_j/\mathcal{M}_{j-1}) \) is finite-dimensional over \( K \) for all \( j \).

Then \( H^0(P^n, \mathcal{O}_F) \) is finite-dimensional.

Proof. By induction on \( s \). If \( s = 1 \), the claim is trivial. So, suppose \( H^0(P^n, \widetilde{\mathcal{M}}_{j-k}) \) is finite-dimensional for all \( k \geq 1 \). We have an exact sequence

\[ 0 \to H^0(\widetilde{\mathcal{M}}_{j-1}) \to H^0(\widetilde{\mathcal{M}}_j) \to H^0(\widetilde{\mathcal{M}}_j/\mathcal{M}_{j-1}) \]

from the exact sequence

\[ 0 \to \widetilde{\mathcal{M}}_{j-1} \to \widetilde{\mathcal{M}}_j \to \mathcal{M}_j/\mathcal{M}_{j-1} \to 0 \]

of sheaves.
By inductive hypothesis the left and right-hand terms are finite-dimensional, thus the middle is as well.

So, we have reduced the theorem to the case \( f = \frac{k[x_0, \ldots, x_n]}{\mathfrak{p}} \).

Note if \( x_i \notin \mathfrak{p} \) then \( \text{of} = 0 \).

Now, suppose \( x_i \notin \mathfrak{p} \). Then for any \( f \), multiplication by \( x_i \) induces an injection \( \left( \frac{k[x_0, \ldots, x_n]}{\mathfrak{p}} \right)_f \hookrightarrow \left( \frac{k[x_0, \ldots, x_n]}{\mathfrak{p}} \right)_f \).

We thus get an injective sheaf map

\[ \mathcal{F} \rightarrow \mathcal{F}(l) \]

by multiplication by \( x_i \) for any \( l \geq 1 \).

Hence \( H^0 (\mathcal{F}) \leq H^0 (\mathcal{F}(l)) \) \( \forall l \geq 1 \).

Finally, by applying our earlier corollary again together with Serre's Theorem, we
\[ H^0(\mathcal{O}(-m+l)^K) \rightarrow H^0(\mathcal{O}J(l)) \]
surjective for \( l \) large. But
\[ H^0(\mathcal{O}(m+l)^K) = (k[x_0, \ldots, x_n]_{m+l})^K, \]
finitely-dimensional, implying that
\[ H^0(\mathcal{O}J(l)) \]
and its subspace \( H^0(\mathcal{O}) \)
are finitely-dimensional as well. \( \square \)

Note: We have turned around the usual order here — Serre's theorem is usually proven later! And the fact that \( J = \overline{J} \) is usually proven using our lemma, not the other way around.

Cor: If \( X \rightarrow Y \) is projective and \( \mathcal{O}_Y \)
a coherent \( \mathcal{O}_X \)-module, then \( \mathcal{O}_X \mathcal{F} \)
a coherent \( \mathcal{O}_Y \)-module.