Localization [see Matsumura or Eisenbud].

$R$ a ring (commutative with 1), $S \subseteq R$ a subset.

$S$ is multiplicative if (a) $1 \in S$, (b) $x, y \in S \Rightarrow xy \in S$.

The localization of $R$ at $S$ (or ring of fractions) is a ring $S^{-1}R$ (or $R_S$) with a homomorphism $R \rightarrow S^{-1}R$ such that any homomorphism $R \rightarrow T$ to a ring $T$ such that $y(s)$ is a unit in $T$ for every $s \in S$, factors uniquely through $l$. 

\[ l \downarrow \]
\[ S^{-1}R \rightarrow T. \]

Exercise $S^{-1}R$ is unique up to canonical isomorphism.

Exercise Read about localization, remind yourself about examples.

Theorem

1) Ideals of $R_S$ are all of the form $I R_S$, $I \subset R$ an ideal.

2) $P R_S$ is prime if and only if $P \subset R$ is a prime with $P N S = \emptyset$.

Special Case of Localization If $P \subset R$ is prime,

$S = R \setminus P$ is a multiplicative set, and we write $R_P$ for $S^{-1}R$; moreover, $R_P$ is a local ring with unique maximal ideal $P R_P$. 
Affine Schemes

Now we're ready to endow Spec $(R)$ for a ring $R$ with the extra structure we need.

Recall $R$ a ring (always commutative with 1).

$$\text{Spec } (R) = \{ \mathfrak{p} \in R \mid \mathfrak{p} \text{ is a prime ideal} \}.$$ 

We give it the Zariski topology: closed sets are those of the form

$$V(I) = \{ \mathfrak{p} \in \text{Spec } (R) \mid I \subseteq \mathfrak{p} \},$$

where $I \subseteq R$ is an ideal.

Note $V(0) = \text{Spec } (R), \quad V(R) = \emptyset$.

More to the point:

**Lemma**

1. If $I, J \subseteq R$ are ideals,

$$V(IJ) = V(I) \cup V(J).$$

2. If $\{ \mathfrak{a}_\alpha \}_{\alpha \in A}$ is a collection of ideals,

$$V(\bigcap_{\alpha} \mathfrak{a}_\alpha) = \bigcap_{\alpha} V(\mathfrak{a}_\alpha).$$

3. If $I, J \subseteq R$ are ideals,

$$V(I) \subseteq V(J) \iff \sqrt{I} \supseteq \sqrt{J}.$$
Proof. Recall
\[ IJ = \{ \sum_{k} a_k b_k \mid a_k \in I, b_k \in J \}. \]

1. If \( Pe \in V(IJ) \) then \( I \subseteq P \) or \( J \subseteq P \); since
   \( IJ \subseteq I \cap J \), we get \( Pe \in V(IJ) \). Conversely, if \( IJ \subseteq P \) and \( I \not\subseteq P \), choose \( x \in I \) such that \( x \not\in P \). If \( y \in J \) then \( xy \in IJ \subseteq P \), so \( y \in P \) since \( P \) is prime; thus \( I \subseteq P \).

2. If \( Pe \in \bigcap_{x} V(Ix) \) then \( I_x \subseteq P \), so
   \[ \sum_{x} I_x \subseteq P \] as well. Conversely, since \( I_x \subseteq \bigcap_{x} I_x \),
   if \( Pe \in V(\bigcap_{x} I_x) \) then \( Pe \in V(Ix) \) \( \forall x \).

3. \( V(I) \subseteq V(J) \) iff \( (I \subseteq P \Rightarrow J \subseteq P) \). Now
   \[ \sqrt{I} = \bigcap_{P \text{ prime}} P \bigcap_{I \subseteq P} I \]
   iff \( J \subseteq \sqrt{I} \) iff \( I \subseteq \sqrt{I} \). \( \square \)

Remark. We used

Lemma \( \sqrt{I} = \bigcap_{P \text{ prime}} P \bigcap_{I \subseteq P} I \)

Proof. \( \sqrt{I} \) def \( \{ x \in R \mid x^n \in I \text{ for some } n \geq 1 \} \).

If \( x^n \in I \subseteq P \) then \( x \in P \) for \( P \) prime. Conversely, if
\( x \not\in \sqrt{I} \) then \( S = \{ 1, x, x^2, \ldots \} \) is a multiplicative set, \( S \not\subseteq I \).
so $S^{-1} I \subseteq S^{-1} R$ is a proper ideal, and if $S^{-1} m \subseteq S^{-1} R$ is a prime ideal with $S^{-1} I \subseteq S^{-1} m$ then $m = S^{-1} m \cap R$ is a prime ideal containing $I$ but $m \cap S \neq \emptyset$. \[ \]

Cor: Define the open sets of the Zariski topology to be complements to the closed sets $V(I)\setminus \{I \in \text{Spec } R\}$. This is a topology on $\text{Spec } R$.

We next define a sheaf of rings, $\mathcal{O} = \mathcal{O}_{\text{Spec } R}$, on $\text{Spec } R$, called the structure sheaf.

Def: A sheaf $\mathcal{F}$ on a top space $X$ is a sheaf of rings if it comes equipped with a morphism $\mathcal{F} \times \mathcal{F} \rightarrow \mathcal{F}$ of sheaves of abelian groups, such that

1) each $m : \mathcal{F}(U) \times \mathcal{F}(U) \rightarrow \mathcal{F}(U)$ gives $\mathcal{F}(U)$ the structure of a ring, and

2) each $p : \mathcal{F}(U) \rightarrow \mathcal{F}(V)$ is a ring homomorphism.

Given an open set $U \subseteq \text{Spec } R$, let

$$\mathcal{O}(U) = \left\{ s : U \rightarrow \prod_{P \in U} R_P \mid s(p) \in R_p \text{ and for each } p \in U \text{ there are a neighborhood } V_p \text{ of } p \text{ in } U \text{ and } \exists \epsilon \in R \text{ with } h \neq \epsilon \text{ for all } q \in V_p, \text{ such that } s(q) = \frac{1}{\epsilon} \text{ in } R_q \forall q \in V_p \right\}.$$
Lemma \( \mathcal{U} \) is a presheaf of rings.

**Proof.** First, suppose \( s, t \in \mathcal{U}(V) \), and define

\[
(s + t)(P) = s(P) + t(P).
\]

If \( V_P, V'_P \) are neighborhoods of \( P \) in \( U \) with \( s(Q) = \frac{r}{f} \in R_Q, \ t(Q) = \frac{r'}{f'} \in R_Q \)

for all \( Q \in V_P \), resp. \( Q \in V'_P \), then

(1) \( f \neq 0 \) and \( t \neq 0 \) \( \Rightarrow \) \( f t' Q \in V_P \cap V'_P \)

(2) \( (s + t)(Q) = \frac{r}{f} \frac{r'}{f'} = \frac{rr'}{ff'} \) in \( R_Q \) \( \forall Q \in V_P \cap V'_P \)

So \( \mathcal{U}(V) \) is an abelian group, 

(3) \( (st)(Q) = s(Q)t(Q) = \frac{rr'}{ff'} \) in \( R_Q \) \( \forall Q \in V_P \cap V'_P \)

So \( \mathcal{U}(V) \) forms a subring of

\[
\left\{ s : U \rightarrow \bigoplus_{P \in U} R_P \mid s(P) \in R_P \right\}.
\]

It's straightforward to see that

\[
\mathcal{U}(V) \rightarrow \mathcal{U}(V) \text{ defined by}
\]

\[
\begin{array}{c}
s \mapsto s|_V \\
\end{array}
\]

is a ring homomorphism

\[
\text{restriction of the function } U \rightarrow \bigoplus_{P \in U} R_P.
\]

This proves the lemma.
Prop \( \mathcal{O} \) is a sheaf on \( \text{Spec}(R) \).

Proof. Suppose \( U \subseteq \text{Spec}(R) \) is open, \( \{ V_\alpha \} \) an open cover of \( U \), \( s_\alpha \in \mathcal{O}(V_\alpha) \) \( \forall \alpha \), with
\[
s_\alpha |_{V_\alpha \cap V_\beta} = s_\beta |_{V_\alpha \cap V_\beta} \quad \forall \alpha, \beta.
\]
We certainly get a function \( s : U \to \amalg_{\mathfrak{p} \in \mathcal{U}} R_\mathfrak{p} \) by

\[
s(\mathfrak{p}) = s_\alpha(\mathfrak{p}) \quad \text{for any } \alpha \text{ such that } \mathfrak{p} \in \text{Spec}_\alpha.
\]

Given \( \mathfrak{p} \in \text{Spec}_\alpha \), choose \( \alpha \) such that \( \mathfrak{p} \in \text{Spec}_\alpha \), and
\( V_\alpha \subseteq U_\alpha \) and \( r, f \in R \) such that \( s_\alpha(q) = \frac{r}{f} \) in \( R_{V_\alpha} \)
for all \( q \in V_\alpha \). Then certainly \( s(q) = \frac{r}{f} \) for all \( q \in V_\alpha \); so \( s \in \mathcal{O}(U) \), and \( s|_{V_\alpha} = s_\alpha \) \( \forall \alpha \).

Note that if \( s \in \mathcal{O}(U) \) then \( s|_{V_\alpha} = 0 \) iff
\[
s(q) = 0 \quad \text{in } R_{V_\alpha} \quad \text{for all } q \in V_\alpha.
\]
So if \( s|_{V_\alpha} = 0 \) for all \( \alpha \), then \( s(q) = 0 \) in \( R_{V_\alpha} \) for all \( q \in U \), and so \( s = 0 \).

Def. \( \text{Spec}(R) \) is the pair consisting of the topological space
formerly known as \( \text{Spec}(R) \) (with the Zariski topology)
and the sheaf of rings \( \mathcal{O} \) defined above.
Lemma \( f \in R \). Then \( P \not\in V(f) \) if and only if \( \{1, f, f^2, f^3, \ldots \} \cap P = \emptyset \).

Proof. \( P \not\in V(f) \) \iff \( f \not\in P \) \iff \( f^k \not\in P \) for all \( k \geq 0 \). \( \square \)

Corollary Let
\[
D(f) = \{ P \in \text{Spec}(R) \mid 1, f, f^2, \ldots \} \cap P = \emptyset \}
\]
Then \( D(f) \subset \text{Spec}(R) \) is open in the Zariski topology, and it is exactly the image of the map
\[
\text{Spec}(R_f) \to \text{Spec}(R)
\]
\[
P_R \mapsto P = l^{-1}(P_R), \quad l: R \to R_f
\]

A localization homomorphism \( l \) is an isomorphism. \( \square \)

Lemma The sets \( D(f) \), \( f \in R \), form a basis for the Zariski topology of \( \text{Spec}(R) \).

Recall that \( \{U_a \} \) is a basis for a top. space \( X \) if for every open \( U \subseteq X \), \( U = \bigcup U_a \).

Proof of Lemma. Let \( V = \text{Spec}(R) \setminus V(I) \).
If \( P \in U \) then \( I \not\in P \), so there is some \( f \in I \) with \( f \not\in P \); then \( (f) \leq I \) implies \( V(I) \subseteq V(f) \), but \( f \not\in P \) so \( P \in D(f) \). Also \( D(f) \cap V(I) = \emptyset \). \( \square \)
Lemma \( \text{Spec}(R) = \bigcup_{\alpha} D(f_{\alpha}) \) if and only if \( R = (f_{\alpha} \cap \mathbb{R}) [\text{for } \alpha \text{ subset } \mathbb{R} ] \).

Proof. \( \bigcup_{\alpha} D(f_{\alpha}) = \text{Spec}(R) \setminus \bigcap_{\alpha} V(f_{\alpha}) \), so the lemma is equivalent to:
\[
\bigcap_{\alpha} V(f_{\alpha}) = \emptyset \text{ iff } R = (f_{\alpha} \cap \mathbb{R}).
\]

By an earlier lemma,
\[
\bigcap_{\alpha} V(f_{\alpha}) = V((f_{\alpha} \cap \mathbb{R})),
\]
which is empty iff \((f_{\alpha} \cap \mathbb{R})\) is not a proper ideal, i.e. iff \( R = (f_{\alpha} \cap \mathbb{R}) \). □

Prop. For any \( p \in \text{Spec}(R) \),
\[
U_{\text{Spec}(R), p} = Rp.
\]

Proof. Define a homomorphism
\[
U(U) \rightarrow Rp
\]
\[
s \mapsto s(p) \quad \text{for any } p \in V \subseteq \text{Spec}(R).
\]
This clearly satisfies:
\[
(U(U)) \quad \text{commutes for all } p \in V \cup U \setminus \text{Spec}(R).
\]
So we get a homomorphism \( U_p \rightarrow Rp \).
Given \( \frac{r}{t} \in R_P \) with \( r, t \in R \), \( t \neq P \), we can define \( s(Q) = \frac{r}{t} \in R_Q \) \( \forall Q \in D(f) \). Then \( P \mapsto \frac{r}{t} \in R_P \); so \( \Psi_P \) is surjective.

Next, suppose \( \Psi_P(t) = 0 \) for \( t \in Q_P \); then \( t(\mathcal{P}) = 0 \). There is a neighborhood \( U \) of \( p \) and an \( s \in V(U) \) with the image of \( s \) in \( Q_p \) equal to \( t \). By construction of \( \Omega \), there is a neighborhood \( V_p \) of \( p \) in \( U \) and \( r, t \in R \) with \( s_Q = \frac{r}{t} \in R_Q \) for all \( Q \in V_p \). Now \( \frac{r}{t} = 0 \) in \( R_P \), so there is \( g \in R \setminus P \) such that \( gr = 0 \) in \( R \). Then also \( \frac{r}{t} = 0 \) in \( R_Q \) whenever \( g \in R \setminus Q \) and \( Q \in V_p \), i.e., \( s = 0 \) in \( \Omega(D(g) \cap V_P) \) thus \( t = 0 \) in \( Q_P \). \( \square \)

**Prop.** For any \( f \in R \), \( \Omega(D(f)) = R_P = S^{-1}R \) where 
\[
S = \{1, t, t^2, \ldots \}.
\]

**Proof.** Define \( \tilde{\Psi} : R \rightarrow \Omega(D(f)) \) by
\[
\tilde{\Psi}(r) : D(f) \rightarrow \bigcup_{P \in D(f)} R_P
\]
\[
P \mapsto \tilde{\Psi}(P) = r.
\]
Since \( f^k \) is a unit in \( R_P \) for all \( k \geq 1 \) whenever \( f \notin P \), \( \tilde{\Psi} \) factors uniquely through
\[
R_P \xrightarrow{\Psi_P} \Omega(D(f)).
\]
where $\varphi (r f^{-k}) = (p \rightarrow r f^{-k} \in R_p \quad \forall P \in D(f))$.

$\varphi$ is injective. Suppose $\varphi (r f^{-k}) = 0$; then

$r f^{-k} = 0$ in $R_p$ for every $P \in D(f)$. Thus,

for each $P \in D(f)$ there is $h_P \in R$ with $h_P^r = 0$, $h_P \notin P$.

In particular, we find that

$\text{Ann}_R (r) \notin P$ for all $P \in D(f)$,

so $V(\text{Ann}_R (r)) \cap D(f) = \emptyset$

$\Rightarrow V(\text{Ann}_R (r)) \subseteq V((f)) \Rightarrow f \in \sqrt{\text{Ann}_R (r)}$,

i.e., $f^l r = 0$ in $A$ for some $l > 0$. Thus $r f^{-k} = 0$ in $R_f$.

$\varphi$ is surjective

Suppose $s \in U(D(f))$. Cover $D(f)$ by open sets $V_i$ over which $s$ is represented by $a_i / g_i$, $g_i \notin P \quad \forall P \in V_i$, i.e. $V_i \subseteq D(g_i)$. WLOG we may assume $V_i = D(h_i)$ for some $h_i \in R$. Then $D(h_i) \subseteq D(g_i)$, so $V(h_i) \supseteq V(g_i)$, so $\sqrt{h_i} \subseteq \sqrt{g_i}$, i.e. $h_i^m \in (g_i)$ for some $m$. Then $h_i^m = c_i g_i$, and so replacing $a_i / g_i$ by $c_i g_i / h_i^m$ we find that $s$ is represented by $b_i / k_i$ [$b_i = c_i a_i$, $k_i = h_i^m$] on $D(k_i)$ and

[$\text{since } D(h_i) = D(h_i^m)$] the $D(k_i)$ cover $D(f)$. 

Claim: The open cover \( D(f) = \bigcup_i D(k_i) \) has a finite subcover.

Proof. This follows from our lemmas that \( D(f) = \text{Spec}(R_f) \) and, then, that \( \text{Spec}(R_f) = \bigcup_i D(k_i) \Rightarrow (\{k_i\}) = R_f \). We then find that \( \exists k_1, \ldots, k_n \) among our \( k_i \) and \( r_1, \ldots, r_n \in R \) such that \( \sum r_i k_i = f^m \) for some \( m \geq 0 \). Then
\[
V((f^m)) = V((f)) \supseteq V((k_1, \ldots, k_n)) = \bigcap_{i=1}^n V((k_i))
\]
\[
\Rightarrow D(f) \subset \bigcup_{i=1}^n D(k_i).
\]

For \( 1 \leq i, j \leq n \) we have that \( b_i/k_i, b_j/k_j \) both represent \( s \), on \( D(k_i) \cap D(k_j) = D(k_ik_j) \). By the injectivity of \( \Phi \), we get \( b_i/k_i = b_j/k_j \) in \( R_{k_ik_j} \), say with
\[
(k_ik_j)^{n_{ij}} (b_i/k_i - b_j/k_j) = 0 \quad \text{in } R.
\]

Let \( n = \max_{1 \leq i, j \leq n} n_{ij} \). Replace \( k_i \) by \( k_i^{n_{ij}} \), \( b_i \) by \( b_i^{n_{ij}} \). Then \( b_i/k_i \) still represents \( s \), on \( D(k_i) \), but also \( b_i/k_i = b_j/k_j \) \( \forall 1 \leq i, j \leq n \); we also still have \( D(f) = \bigcup_i D(k_i) \), so there are \( x_1, \ldots, x_n \) and \( m \) such that \( \sum x_i k_i = f^m \). Set \( r = \sum x_i b_i \). Then \( k_j r = \sum x_i b_i k_j = \sum x_i k_i b_j = b_j f^m \), so
\[
\frac{r}{f^m} = \frac{b_j}{k_j} \quad \text{on } D(k_j).
\]
Thus \( \Phi(\frac{r}{f^m}) = s. \]
Cor: $\mathcal{O}(\text{Spec } R) = R$.

**Notation**: It is customary to write $\Gamma(\text{Spec } R, \mathcal{O})$ or $H^0(\text{Spec } R, \mathcal{O})$ for $\mathcal{O}(\text{Spec } R)$.

[Or in general, for a sheaf $\mathcal{F}$ on $X$,

$$\Gamma(X, \mathcal{F}) = H^0(X, \mathcal{F}) = \mathcal{F}(X)$$]
Def A ringed space is a pair \((X, \mathcal{O}_X)\) consisting of a topological space \(X\) and a sheaf of rings \(\mathcal{O}_X\) on \(X\). A morphism of ringed spaces \((X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)\) consists of a continuous map \(f: X \rightarrow Y\) and a homomorphism \(f^\#: \mathcal{O}_Y \rightarrow \mathcal{O}_X\) of sheaves of rings on \(Y\).

Def A ringed space \((X, \mathcal{O}_X)\) is a locally ringed space if for each \(p \in X\), the stalk \(\mathcal{O}_{X, p}\) is a local ring. A morphism \((X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)\) of ringed spaces that are also locally ringed spaces will be assumed to satisfy: for all \(p \in X\),

\[
(\mathcal{O}_Y, f(p)) = \lim_{\substack{\longrightarrow \\ f(p) \in V}} \mathcal{O}_Y(V) \rightarrow \lim_{\substack{\longrightarrow \\ f(p) \in V}} \mathcal{O}_X(f^{-1}(V))
\]

satisfies \((\mathcal{O}_Y, f(p)) = \mathcal{O}_{X,p}\), where \(\mathcal{O}_{X,p} \subseteq \mathcal{O}_X, p\) are the maximal ideals.
Cor (Spec(R), O_{Spec(R)}) is a locally ringed space.

Def. A locally ringed space (X, O_X) is an affine scheme if (X, O_X) is isomorphic, as a locally ringed space, to (Spec(R), O_{Spec(R)}) for some (commutative with 1) ring R.

Prop. If \( \varphi : R \rightarrow S \) is a homomorphism of rings (commutative with 1), then \( \varphi \) induces a natural morphism of locally ringed spaces \( (f, f^\#) : (\text{Spec}(S), O_{\text{Spec}(S)}) \rightarrow (\text{Spec}(R), O_{\text{Spec}(R)}) \).

The construction determines a \( \text{Comm Rings} \) \( ^P \rightarrow (\text{Affine Schemes}) \).

Moreover, the induced map
\[
f^\#: \text{Spec}(R) = \text{Spec}(S) \rightarrow \text{Spec}(R) \hookrightarrow \text{Spec}(S)
\]
is \( \varphi \).

Proof. Given \( \varphi \), define
\[
f^\#: \text{Spec}(S) \rightarrow \text{Spec}(R) \text{ by } P \mapsto \varphi^{-1}(P) = \varphi(P).
\]

If \( I \subseteq R \) is an ideal, then
\[
f^{-1}(V(I)) = \{ P \subseteq S \mid \varphi^{-1}(P) \not\subseteq I \}
\]
\[
= \{ P \subseteq S \mid P \supseteq \varphi(I) \cdot S \} = V(\varphi(I)S).
\]
So \( f \) is continuous. For each \( P \in \text{Spec}(S) \), we get a local homomorphism
\[
R(P^{-1}(P)) \to S_P.
\]

Now, define, for \( V \subseteq \text{Spec}(R) \) open, a map
\[
\{ s : V \to \coprod_{Q \in V} R_Q | \forall a \in R_Q, s(a) \in R_{Q} \} \overset{\tilde{f}^{\#}}{\longrightarrow} \{ t : f^{-1}(V) \to \coprod_{P \in f^{-1}(V)} S_P | t(p) \in S_P \}
\]
as follows, let
\[
\tilde{f}^{\#}(s)(P) = \phi_{P}^{S}(s(P^{-1}(P))).
\]

Note that
\[
\tilde{f}^{\#}(\mathcal{O}(V)) \subseteq \phi_{\mathcal{O}}^{S}(\text{Spec}(S))(V) = (\mathcal{O}_{\text{Spec}(S)})(f^{-1}(V)).
\]

Indeed, if \( W \subseteq V \) is open and \( s \in \mathcal{O}(V) \) is of the form \( \frac{r}{g} \) on \( W \), and \( P \in f^{-1}(W) \), then \( \frac{r}{g} \in \phi_{P}(P^{-1}(P)) \Rightarrow \phi_{P}(s) \in \mathcal{O}(W) \Rightarrow \phi_{P}(s) \) defines a section of \( \mathcal{O}_{\text{Spec}(S)} \) over \( f^{-1}(W) \), and its image in each \( S_P \), \( P \in f^{-1}(W) \), agrees with \( \tilde{f}^{\#}(s)(P) \), since
\[
\begin{array}{ccc}
R_{P} & \overset{\phi_{P}}{\longrightarrow} & S_{P} \\
\downarrow & & \downarrow \\
R(P^{-1}(P)) & \overset{\phi_{P}}{\longrightarrow} & S_{P}
\end{array}
\]
commutes.

It follows that \( \tilde{f}^{\#}(\mathcal{O}(W)) \) determines a morphism of sheaves of rings
\[
\tilde{f}^{\#} : \mathcal{O}_{\text{Spec}(R)} \to \mathcal{O}_{\text{Spec}(S)}.
\]
The map on stalks

\[ f^*_P : \mathcal{O}_{\text{Spec}(R), f^*(P)} \to \mathcal{O}_{\text{Spec}(S), P} \]

is clearly \( \varphi_P : R_{\varphi^{-1}(P)} \to S_P \)

by our description of the stalks on \( \text{Spec}(R), \text{Spec}(S) \).

Hence \((f, f^*)\) is a morphism of locally ringed spaces.

It is clear that \( \varphi \mapsto (f, f^*) \) is functorial in \( f \).

It immediately follows from our construction of \( f^* \) that these maps are functorial in \( f \), hence so is \( f^* \).

Finally, it follows from our construction/description of 

\( \mathcal{O}_{\text{Spec}(R)}(\text{Spec}(R)) = R \) and \( f^* \mathcal{O}_{\text{Spec}(S)}(\text{Spec}(R)) = S \)

that \( f^* \mathcal{O}_{\text{Spec}(R)} \) is identified with \( \varphi^* \); indeed, for

every open set \( U \subseteq \text{Spec}(R) \) we get a commutative diagram

\[
\begin{array}{ccc}
R & \xrightarrow{\varphi} & S \\
\downarrow & & \downarrow \\
\mathcal{O}_{\text{Spec}(R)}(U) & \xrightarrow{f_*} & \mathcal{O}_{\text{Spec}(S)}(U) = \mathcal{O}_{\text{Spec}(S)}(f^{-1}(U))
\end{array}
\]

The vertical maps are isomorphisms when \( U = \text{Spec}(R) \)

by an earlier proposition. \( \square \)
Prop. Any morphism of locally ringed spaces

\[
(Spec(S), O_{Spec(S)}) \xrightarrow{(f, f^\#)} (Spec(R), O_{Spec(R)})
\]

is induced by a homomorphism of rings

\[ \psi: R \to S. \]

Proof. Let

\[
\psi = f^\#: Spec(R) \to Spec(S) \quad \text{Spec(R)} \to f^\#(O_{Spec(S)}(\text{Spec R}) \quad \text{Spec(S)}
\]

\[ R \quad S \]

For every \( P \in Spec(S) \) we get a commutative diagram

\[
\begin{array}{ccc}
R & \xrightarrow{\psi} & S \\
\downarrow & & \downarrow \\
(O_{Spec(R)}, f(P)) & \xrightarrow{f^\#} & (O_{Spec(S)}, \mathfrak{p}_P) \\
\downarrow & & \downarrow \\
R_{f(P)} & \xrightarrow{\mathfrak{p}_P} & S_{P}
\end{array}
\]

implying that \( f(P) = \psi^{-1}(P) \). Since the map \( f^\# \) on sheaves is completely determined by the collection

\[ f^\#: \mathfrak{p}_P \to O_{Spec(S)}, \]

this completes the proof. \( \square \).
Corollary The functor
\[(\text{Comm Rings})^\text{op} \rightarrow (\text{Affine Schemes})\]
is an equivalence of categories.

Proof. The last two propositions proved that it was faithful and full, respectively. It's essentially surjective by definition.

Def A locally ringed space \((X, \mathcal{O}_X)\) is a scheme if for every \(p \in X\) there is an open set \(U \subseteq X\) containing \(p\) such that \((U, \mathcal{O}_X|_U)\) is an affine scheme.

As with other kinds of spaces, e.g., topological spaces or Riemannian manifolds, there are local structures — for example, curvature — and global structures (for example, the fundamental group). We're going to think about local structures for awhile, so we may as well mostly restrict our attention to affine schemes.