DVRs

A ring $R$ is a *discrete valuation ring* or *DVR* if it is a domain and its fraction field $K(R)$ comes equipped with a valuation $v: K(R)^* \to \mathbb{Z}$ satisfying

1. $v(xy) = v(x) + v(y)$ \quad \forall x, y \in K(R)^*$.
2. $v(x + y) \geq \min(v(x), v(y))$ \quad \forall x, y \in K(R)^*$.
3. $v(K(R)^*) = \mathbb{Z}$.

**Lemma** Let $R$ be a DVR. Then

1. $R$ is a local ring.
2. $R$ is 1-dimensional.
3. Every nonzero ideal of $R$ is of the form $m^k$ for some $k \geq 1$, where $m$ is the maximal ideal of $R$.
4. $\dim_{R/m}(m/m^2) = 1$.
5. The valuation $v$ on $K(R)^*$ is unique provided $v(K(R)^*) = \mathbb{Z}$. 
Proof. Note that
\[ v(y) = v(1 - y) = v(1) + v(y) \quad \forall y \in K(R)^{x} \]
\[ \Rightarrow v(1) = 0. \text{ Hence } \exists \gamma \in K(R)^{x}, \]
\[ 0 = v(1) = v(yy^{-1}) = v(y) + v(y^{-1}) \]
\[ \Rightarrow v(y^{-1}) = -v(y). \text{ Thus } x \in R \setminus \{0\} \]
is a unit of \( R \), written \( x \in R^{\times} \), iff \( v(x) = 0 \).

Let \( m = \{y \in R | x = 0 \text{ or } v(x) \geq 1/2 \} \).

Then \( v(yx) = v(y) + v(x) \geq v(x) \quad \forall y \in R \setminus \{0\} \),
and \( v(x + x') = \min(v(x), v(x')) \geq 0 \quad \forall x, x' \in m. \)

So \( m \) is an ideal of \( R \). But also \( m = R \setminus R^{\times} \), so it is the union of
the maximal ideals of \( R \) (an element
lies in some maximal ideal iff it is not
a unit). Thus \( m \) is the unique
maximal ideal of \( R \).

\[ \]
(2,3). Choose any $\pi \in \mathfrak{m} \setminus \mathfrak{m}^2 = \{x \in R \mid \nu(x) = 1\}$.

If $0 \neq y \in \mathfrak{m}$, say $\nu(y) = k$, then

$$\nu(y \pi^{-k}) = \nu(y) - k \nu(\pi) = 0 \text{ in } k(R),$$

so $u = y \pi^{-k}$ is a unit of $R$, and

$y = u \pi^{k}$ in $R$. Suppose $0 \neq I \subseteq \mathfrak{m}$

is a nonzero proper ideal, and let

$k = \min \{ n \mid \nu(x) = n \text{ for some } x \in I \setminus \{0\} \}$.

If $y \in I$ and $\nu(y) = k$, we get

$y = u \pi^{k}$ for some $u \in R^x$, so $\pi^k \in I$.

Thus $(\pi^k) = \mathfrak{m}^k \subseteq I$. But now

every $y \in I \setminus \{0\}$ is of the form $y = u \pi^{k+l}$

for some $u \in R^x$ and $l \geq 0$, so $I = \mathfrak{m}^k$.

This proves (2) and (3).

(4) We have a map of $R/\mathfrak{m} - \text{modules}

$$R/\mathfrak{m} \to \mathfrak{m}/\mathfrak{m}^2 = (\pi)/\pi^2$$
given by \( f \mapsto f \cdot \pi \).

If \( f \cdot \pi = 0 \) in \( \mathbb{N}/m^2 \) then
\[ f\pi \in (\pi^2) \, \text{, so } v(f) \geq 1 \]
and \( f \in M \). Hence
\[ R/m \to \mathbb{N}/m^2 \text{ is injective.} \]

Since \( m = (\pi) \), it is surjective as well, hence an isomorphism.

(5) We have seen that for every \( \pi \in m^2 \),
\[ v(\pi) = 1 \], assuming only that the
image of \( v \) is \( \mathbb{Z} \). It is immediate
from this and the fact that
\[ v(w) = 0 \text{ for } w \in R^x \text{ (also proven above) } \]
that \( v(x) \) is uniquely determined for all
\( x \in R \setminus \{0\} : v(x) = k \text{ if } x \equiv m^k \mod m^k \).

Since every element of \( K(R)^{x} \) is a fraction
from \( R \), uniqueness follows by DVR axiom
(1). \( \square \)
Def A local ring \((R, m)\) of dimension \(d\) is regular if \(\dim_{R/m}(m/m^2) = d\).

Cor A DVR is a regular local ring.

Facts (1) A regular local ring is a domain.
(2) If \(R\) is a regular local ring and \(P\) is a prime ideal of \(R\), then \(R_P\) is regular.
(3) A 1-dimensional regular local ring is a DVR.

Def A regular ring is a noetherian ring \(R\) such that every localization \(R_P\) at a prime \(P\) of \(R\) is a regular local ring. [Equivalently, by fact (2), at every maximal ideal of \(R\)].
Jacobian Criterion for Regularity

\[ S = k[x_1, \ldots, x_r], \quad k \text{ a field.} \]

\[ I = (f_1, \ldots, f_s) \subset S, \quad R = S/I. \]

\[ I \subsetneq P \subsetneq S, \quad P \text{ a prime, } K \overset{def}{=} K(R/P). \]

Let \( c = \text{codim } (IS_P, IS_P) = \text{height } (IS_P). \)

Then

1) The Jacobian matrix

\[ J = (\frac{\partial f_i}{\partial x_j}) \mod P \quad \text{(that is, as a matrix of elements of } R/P) \]

has rank at most \( c. \)

2) In case \( \text{char}(k) > 0, \) we assume in addition that \( K/k \) is a separable extension. \( R_P \) is a regular local ring iff \( J \) has rank \( c \) as a matrix of elements of \( R_P/PR_P. \)
Def A variety (separated scheme of finite type) over $k$ is nonsingular if each local ring $O_{X, x}$ is regular for every $x \in X$.

Note $O_{X, x}$ regular $\implies O_{X, x}$ integral.

This implies that $X$ is reduced.

Prop Suppose that $X$ is a noetherian scheme. Suppose $O_{X, x}$ is integral for all $x \in X$. If $X$ is connected then $X$ is irreducible.

Proof sketch. We first prove that if $x \in X$ there exists an open set $U \subseteq X$ s.t. $x \in U$ and $U$ is irreducible. Indeed, choose any affine open $U = \text{Spec}(R)$ containing $x$, $x = \mathfrak{p} \in \text{Spec}(R)$. By hypothesis, $R_{\mathfrak{p}}$ is a domain. Let $Q_1, \ldots, Q_s$ be the associated
primes of \( R \) — these are the annihilators of nonzero elements of \( R \). Then \( U \mathfrak{q}_i \) is the set of zero divisors of \( R \). Since \( R \) is noetherian, there are only finitely many associated primes.

Since \( R \mathfrak{p}_i \) is a domain, we have

\[ \mathfrak{q}_i R \mathfrak{p}_i = 0. \]

Since \( R \) is noetherian, \( \mathfrak{q}_i \) is finitely generated, say \( \mathfrak{q}_i = (f_1, \ldots, f_k) \).

Then \( \exists \, r_1, \ldots, r_k \in R \mathfrak{p}_i \) with

\[ f_i r_j = 0, \quad \text{so} \quad \mathfrak{q}_i R r_1 \cdots r_k = 0 \quad \text{in} \quad R. \]

Taking the product of all such elements over all \( \mathfrak{q}_i \), there is \( r \in R \mathfrak{p}_i \) such that

\[ \mathfrak{q}_i R r = 0 \quad \text{in} \quad R r + \mathfrak{q}_i. \]

One can then check that \( R_r \) is a domain (all the zero divisors of \( R \) are already zero in \( R_r \)). Thus \( x \in \text{Spec}(R_r) \) with \( R_r \) irredu.

Now, suppose \( X = \bigcup_{i=1}^s V_i \) with each \( V_i \) open and irreducible, and suppose
X is connected. Suppose \( X = X_1 \cup X_2 \) with each of \( X_1, X_2 \) proper and closed in \( X \). Then for each \( i \), \( V_i = (V_i \cap X_1) \cup (V_i \cap X_2) \Rightarrow \) either \( V_i \cap X_1 = V_i \) or \( V_i \cap X_2 = V_i \) (since otherwise \( V_i \) is not irreducible).

Now, we recursively modify \( X_1 \) and \( X_2 \):

**Step 0** Set \( X_1^0 = X_1 \), \( X_2^0 = X_2 \).

Note that \( X_1^0 \not\subseteq X_2^0 \), \( X_2^0 \not\subseteq X_1^0 \).

Note also that \( V_i \), either \( V_i \subseteq X_1^0 \) or \( V_i \subseteq X_2^0 \).

**Step i** Suppose \( X = X_1^{i-1} \cup X_2^{i-1} \) a union of closed subsets, with \( X_1^{i-1} \subseteq X_1^{i-1} \), \( X_2^{i-1} \subseteq X_2^{i-1} \), \( X_1^{i-1} \not\subseteq X_2^{i-1} \), \( X_2^{i-1} \not\subseteq X_1^{i-1} \).

Suppose also that for each \( j \).
either \( V_j \subseteq X_1^{i-1} \) or \( V_j \subseteq X_2^{i-1} \).

Suppose also that for \( j \leq i-1 \),
\[
V_j \subseteq X_1^{i-1} \Rightarrow V_j \cap X_2^{i-1} = \emptyset
\]
and \( V_j \subseteq X_2^{i-1} \Rightarrow V_j \cap X_1^{i-1} = \emptyset \).

Now, define \( X_1^i, X_2^i \) as follows: if
\[
V_i \subseteq X_1^{i-1}, \text{ set } X_1^i = X_1^{i-1}, \text{ and }
\]
\( X_2^i = X_2^{i-1} \setminus (V_i \cap X_2^{i-1}) \); otherwise,
set \( X_2^i = X_2^{i-1} \) and \( X_1^i = X_1^{i-1} \setminus (V_i \cap X_1^{i-1}) \).

**Claim:** \( X_1^i, X_2^i \) satisfy:

(a) \( X = X_1^i \cup X_2^i \), and both are closed in \( X \).
(b) \( X_1^i \not\subseteq X_2^i \) and \( X_2^i \not\subseteq X_1^i \).
(c) \( \forall j, V_j \subseteq X_1^i \) or \( V_j \subseteq X_2^i \).
(d) For \( j \leq i \), \( V_j \subseteq X_1^i \Rightarrow V_j \cap X_2^i = \emptyset \); and
\( V_j \subseteq X_2^i \Rightarrow V_j \cap X_1^i = \emptyset \).
Proof of Claim. $X_i^c$ and $X_{-i}^c$ are certainly closed in $X$, since $X_{-k}^c \setminus (X_{-k}^c \cap V_i)$ is closed in $X_{-k}^c$. By construction, we still have $X = X_1^c \cup X_2^c = X_1^c \cup X_2^c$.

For (a), assume WLOG that $V_i \subseteq X_{-1}^c$ and $X_2^c = X_2^c \setminus (V_i \cap X_2^c)$.

If $X_2^c \subseteq X_1^c$ then

$$X_2^c = X_2^c \cup (V_i \cap X_2^c) \subseteq X_1^c \cup V_i = X_1^c,$$

a contradiction. So $X_2^c \not\subseteq X_1^c$. Similarly one gets $X_1^c \not\subseteq X_2^c$.

For (c), we have that $X = X_1^c \cup X_2^c$ a union of proper closed subsets, so

$$V_0 = (V_1 \cap X_1^c) \cup (V_j \cap X_2^c)$$

a union of closed subsets. Since $V_j$ is irreducible by hypothesis, either $V_j \cap X_1^c$ or $V_j \cap X_2^c$ fails to be a proper subset.
Finally, for (d), note that
\[ X_i \subseteq X_{i-1}, \quad X_2 \subseteq X_1, \quad \text{so,} \]
since (d) holds in step (i-1) for all \( j \leq i-1 \),
we need only check for \( j = i \) in step i.
But now \( X_1, X_2 \) are exactly constructed
so (d) holds for \( V_i \).

Finally, then, after step \( s \) is completed we have \( X = X_1 \cup X_2 \) a union of proper closed subsets, and for each \( j \), either \( V_j \subseteq X_1 \) and \( V_j \cap X_2 = \emptyset \), or \( V_j \subseteq X_2 \) and \( V_j \cap X_1 = \emptyset \).

Hence each of \( X_1, X_2 \) is a union of open subsets of \( X \); thus both \( X_1 \) and \( X_2 \) are open and closed in \( X \); a contradiction (\( X \) is connected!) unless one of \( X_1, X_2 \) is empty.

This completes the proof. \( \square \)

**Example** \( R = \mathbb{C}[x,y]/(y^2 - x^3), \)
\( J = (-3x^2, 2y). \) For \( m = (x-a, y-b), \)
\( (y^2 - x^3) \subseteq m \iff b^2 = a^3. \)

Note \( c = \text{ht}(y^2 - x^3) = 1. \)
\( J \) has rank 1 mod \( m \) if \( -3a^2 \neq 0 \) or \( 2b \neq 0. \)
-3a^2 \neq 0 \text{ or } 2b \neq 0.

So \( R(w,y) \) is \underline{not regular}, but \( \text{ Rin } \) is regular for all other maximal \( \mathcal{M} \).
Now we're going to define differential forms for a variety. First we give the horribly general definition.

Let $f : X \to Y$ be a separated morphism of schemes (this hypothesis is unnecessary but it simplifies things for us).

$\Delta : X \to X \times_Y X$ is a closed immersion, so "$\Delta(X)$" is a closed subscheme of $X \times_Y X$. Let $I_\Delta \subseteq \mathcal{O}_{X \times_Y X}$ denote the corresponding quasicoherent sheaf of ideals. Then $\mathcal{O}_{X \times_Y X}/I_\Delta = \Delta^* \mathcal{O}_X$.

**Def.** The sheaf of relative Kähler differentials is $\Omega^1_{X/Y} = I_\Delta/I_\Delta^2$.
Note that $\Omega^1_{X/Y}$ is, by construction, a sheaf of $\mathcal{O}_{X,Y} / I_\Delta$-modules.

Thus, it's naturally a sheaf of $\mathcal{O}_X$-modules. If we pull it back to $X$ via $\Delta$, it gives a quasicoherent sheaf of $\mathcal{O}_X$-modules, also written $\Omega^1_{X/Y}$, such that $\Delta_\ast \Omega^1_{X/Y} \cong I_\Delta / I_\Delta^2$.

**Example** $X = \text{Spec}(S)$, $Y = \text{Spec}(R)$,

$\Delta : \text{Spec}(S) \to \text{Spec}(S \otimes_R S)$ determined by

\[ S \quad \otimes_R S \]

\[ u \otimes \nu \quad \leftarrow \quad u \otimes \nu \]

Then $I_\Delta = \left\{ \sum l_i \otimes v_i \mid \sum u_i \otimes v_i = 0 \text{ in } S \right\}$. 
Then $I_A/I_A \cdot I_A$ is an $S_0 \otimes_R S$-module but, since $I_A$ acts by zero on the quotient module, the action naturally factors through $S_0 \otimes_R S / I_A \cong S$, i.e. $I_A/I_A^2$ is an $S$-module.

Example: $X$ a variety over an algebraically closed field $k$; suppose for simplicity that $X = \text{Spec}(R)$ is affine. Consider the exact sequence

$$0 \to I_A \to R \otimes_R k \to R \to 0$$

coming from $\Delta : X \to X \times_k X$.

We consider it as a sequence of $R$-modules via the left structure. Fix a maximal ideal $m \subset R$; so $R/m = k$ ( $R$ is a finitely generated $k$-algebra by our
assumption on \( X \). Then

\[
\text{Tor}_i^R(R, R/m) = 0 \quad \forall i > 0, \quad \text{so we get, by tensoring with } R/m \text{ under the left } R\text{-action,}
\]

\[
0 \rightarrow R/m \otimes \frac{I}{\Delta} \rightarrow R/m \otimes \frac{R}{\Delta} \rightarrow R/m \rightarrow 0
\]

is exact. Thus \( R/m \otimes \frac{I}{\Delta} = m \). as a right \( R\)-module,

Now \( I_{\Delta} / I_{\Delta}^2 = I_{\Delta} \otimes \frac{R}{\Delta} \otimes \frac{R}{\Delta} / I_{\Delta} \) / so

\[
R/m \otimes \left( I_{\Delta} / I_{\Delta}^2 \right) = I_{\Delta} \otimes \frac{R}{\Delta} \otimes \left( \frac{R}{\Delta} \otimes \frac{R}{\Delta} \right) \otimes \left( \frac{R}{\Delta} \otimes \frac{R}{\Delta} \right)
\]

\[
= I_{\Delta} \otimes \frac{R}{\Delta} \otimes \left( \frac{R}{\Delta} \otimes \frac{R}{\Delta} \right) \otimes \left( \frac{R}{\Delta} \otimes \frac{R}{\Delta} \right)
\]

\[
= m \otimes R/m = m/m^2,
\]

\[
\text{Summary } R/m \otimes \Omega^1_\Delta \otimes \frac{R}{\Delta} / \frac{R}{\Delta} = m/m^2 .
\]
Recall that the tangent space to $\text{Spec}(R)$ at $m$ is $\text{Hom}_{R/m}(R/m, R/m)$. So, the fiber $R/m \otimes_{R} \mathcal{O}_{X/k}^1$ is the cotangent space to $X$ at $m$!

**Fact** If $X$ is a nonsingular $k$-variety, then $\mathcal{O}_{X/k}^1$ is a locally free sheaf.

**Example** $X = \mathbb{A}^n_k = \text{Spec } k[x_1, \ldots, x_n]$.

Have $I_A = (\{ x_1 \partial_{x_1} - 1 \partial_{x_i} \mid i = 1, \ldots, n \})$.

Let $d x_i \overset{df}{=} x_i \partial_{x_i} - 1 \partial_{x_i} + I_A \subset I_A / I_A^2$.

Then one can check that

$\mathcal{O}_{\mathbb{A}^n_k/k}^1$ is a free $\mathcal{O}_{\mathbb{A}^n_k/k}$-module associated to

$k[x] dx_1 \oplus k[x] dx_2 \oplus \ldots \oplus k[x] dx_n$. 
Calculus of forms

$S$ an $R$-algebra. Then

$\Omega_{S/R}$ is the $S$-module generated by the set $\{d(f) | f \in S\}$ modulo relations

\[
\begin{align*}
    d(ab + a'b') &= ad(b) + a'd(b') & \forall a, a' \in R, b, b' \in S \\
    d(bb') &= bd(b') + d(b) \cdot b' & \text{(Leibniz rule)} \\
    & \forall b, b' \in S.
\end{align*}
\]

Fact. If $\text{Spec}(S) \to \text{Spec}(R)$ is a morphism of affine schemes, then

\[
\Omega^{1}_{S/R} = \Omega^{1}_{\text{Spec}(S)/\text{Spec}(R)}
\]

via

\[
\Omega^{1}_{S/R} \to H^{0}(\text{Spec}(S), \Omega^{1}_{\text{Spec}(S)/\text{Spec}(R)}) \xrightarrow{\pi} \frac{I_{\Delta}/I_{\Delta}^{2}}{I_{\Delta}/I_{\Delta}^{2}} \quad d(f) \mapsto \text{foo} f - 1 \text{foo}^{2}.
\]
General example  How to calculate $L^1_{S/R}$ when $S = R[x_1, \ldots, x_r]/I$ ?

Write $I = (f_1, \ldots, f_s)$. Define a map

$$\oplus_i \text{Se}_i \rightarrow \oplus_j S \text{d}x_j$$

by

$$J(e_i) = \sum \frac{\partial f_i}{\partial x_j} \text{d}x_j \quad \text{in} \quad J$$

in other words, $J$ is given by the Jacobian matrix $J = (\partial x_i/\partial x_j)$ regarded as a map of free $S$-modules. Then

$$L^1_{S/R} = \text{coker}(J)$$

[See Eisenbud, Section 16.1.]

Example  $S = R[x]/(f)$. Then

$$L^1_{S/R} = S \text{d}x/(f'(x) \text{d}x) \cong S/(f'(x)).$$
Example \( R = \mathbb{C}, \ S = \mathbb{C}[x,y] / (y^2 - x^3) \).

Then \( \Omega^1_S / \alpha = \text{Sd}x \oplus \text{Sd}y / (2ydy - 3x^2dx) \).

Exercise Check that the fiber of this module over maximal we \text{Spec}(S), i.e. \( \Omega^1_S / \alpha \otimes S / \mathfrak{m} \), is 1-dimensional if \( \mathfrak{m} \neq (x,y) \).

Canonical Sheaf Suppose \( X \) is a nonsingular variety over \( k \). Then \( \Omega^1_{X/k} \) is a locally free \( \mathcal{O}_X \)-module of rank \( d = \dim(X) \). The canonical sheaf (or, by abuse of language, canonical line bundle) of \( X \) is \( K_X \overset{\text{df}}{=} \Lambda^d \Omega^1_{X/k} \).
The canonical sheaf $K_X$ is an invertible sheaf on $X$ (here we are using that $X$ is nonsingular!).

**Example** The canonical sheaf of $\mathbb{A}^n_k$ is a trivial invertible sheaf associated to the module $k[x_1, \ldots, x_n]dx_1 \ldots dx_n$.

**Example** If $X$ is a nonsingular curve (i.e. has dimension 1), then $K_X = \omega_X(X)$. 

**Example** Let $X = \mathbb{P}^1_k$. Writing $\mathbb{P}^1_k = \text{Proj } k[x, y]$, we get

$U_0 = D_+(x) \cong \text{Spec } k\left[\frac{y}{x}\right] = \text{Spec } k[x^2]$.  
$U_1 = D_+(y) \cong \text{Spec } k\left[\frac{x}{y}\right] = \text{Spec } k[y^2]$.  

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A section of $\Omega^1_{\mathbb{P}^1_k/k}$ over $U_0$ is given by $f(z)dz$, $f(z) \in k[z]$. Restricting to $U_0 \cap U_1 = \text{Spec } k[z, z^{-1}]$, we may write $f(z)dz$ in terms of the restriction of the section

$$dz^{-1} \in H^0(U_1, \Omega^1_{\mathbb{P}^1/k})$$

to $U_0 \cap U_1$:

$$dz^{-1} = d\left(\frac{1}{z}\right) = -\frac{1}{z^2}dz$$

Note: The calculus of Kähler differentials is just like that for forms in differential geometry! Read about it in Hartshorne and/or Eisenbud.

So $f(z)dz = -z^2f(z)dz^{-1}$.

Note that we never have $-z^2f(z)dz^{-1} \in C[z^{-1}]dz^{-1}$ unless
f = 0, so we conclude:

Lemma \[ H^0 \left( \mathbb{P}^1_k, \Omega^1_{\mathbb{P}^1_k/k} \right) = 0. \]

Exercise

Use the trivializations

\[ \Omega^1_{\mathbb{P}^1_k | U_1} \cong k[z, z^{-1}] dz^{-1} \]

\[ \Omega^1_{\mathbb{P}^1_k | U_0} \cong k[z] dz \]

to compute the transition functions for the canonical sheaf \( k_{\mathbb{P}^1} = \Omega^1_{\mathbb{P}^1_k/k} \) on \( \mathbb{P}^1 \); use this to prove that

\[ k_{\mathbb{P}^1} \cong \mathcal{O}(-2). \]