Problem 1. Compute $\det \begin{pmatrix} 1 & 1 & 0 \\ 3 & 5 & 7 \\ 2 & 8 & 17 \end{pmatrix}$ by cofactor expansion along the first row.

Answer. $-8$, I believe.

Problem 2. Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation for which $T(1, 1) = (2, 4, 5)$ and $T(-1, 2) = (1, 5, 7)$. Compute $T(4, 2)$.

Answer. $(4, 2) = \frac{10}{3}(1, 1) - \frac{2}{3}(-1, 2)$. Thus $T(4, 2) = \frac{10}{3}(2, 4, 5) - \frac{2}{3}(1, 5, 7) = (6, 10, 12)$. Hopefully my arithmetic is correct!

Problem 3. Suppose $L_A : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation associated to the matrix $A$ with determinant 3.

1) Let $S = \{(x, y) \mid 0 \leq x, y \leq 1\}$. Prove that $L_A(S)$ is a parallelogram in $\mathbb{R}^2$.

2) Compute the area of the parallelogram $L_A(S)$.

Answer. The parallelogram with sides given by the vectors $v, w$ can be defined to be $T = \{t_1v + t_2w \mid 0 \leq t_1, t_2 \leq 1\}$. Now if the columns of $A$ are $v$ and $w$ then $L_A(t_1, t_2) = t_1v + t_2w$; so $L_A(S) = \{L_A(u) \mid u \in S\} = \{t_1v + t_2w \mid 0 \leq t_1, t_2 \leq 1\}$.

Now we have discussed that if $\det(A) = 3$ then the area of the parallelogram $T$ defined by the columns of $A$ has area 3.

Problem 4. State the Dimension Theorem.

Answer. Look it up if you don’t know it!

Problem 5. Compute a basis of the null space of the matrix $A = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 2 & -1 \end{pmatrix}$.

Answer. We have that $A$ is row equivalent to $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ in RREF. Thus the null space is given by $\{(-3t, \frac{7}{2}t, t) \mid t \in \mathbb{R}\}$, which has basis given by the single vector $(-3, \frac{7}{2}, 1)$.

Problem 6. Find the rank of $B = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

Answer. This matrix is row equivalent to $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0.5 \\ 0 & 0 & 0 \end{pmatrix}$, which has two pivots, therefore rank 2.

Problem 7. Consider the linear transformation $T : \mathbb{R}[x]_{\leq 2} \to \mathbb{R}[x]_{\leq 2}$ given by $T(p(x)) = x^2 \frac{dp}{dx} + x \frac{dp}{dx} + p(0)$. Compute the matrix of $T$ with respect to the basis $\beta = \{1, x, x^2\}$.

Answer. $T(1) = 1$, $T(x) = x$, $T(x^2) = 4x^2$. Thus

$$[T]_\beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$
**Problem 8.** Consider bases $\beta = \{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}$ and $\beta' = \{(0, 2, 0), (1, 0, 1), (0, 0, 1)\}$. Compute the change-of-coordinate matrix $Q$ that changes $\beta'$ coordinates into $\beta$ coordinates.

*Answer.* $(0, 2, 0) = -2(1, 0, 0) + 2(1, 1, 0) + 0(1, 1, 1)$, $(1, 0, 1) = (1, 0, 0) - (1, 1, 0) + (1, 1, 1)$, $(0, 0, 1) = 0(1, 0, 0) - (1, 1, 0) + (1, 1, 1)$. Thus

$$[I_{\mathbb{R}^2}]_{\beta'}^\beta = \begin{pmatrix} -2 & 1 & 0 \\ 2 & -1 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$

**Problem 9.** Suppose that $A$ is an $n \times n$ matrix that satisfies $A^2 = 0$ (the zero matrix). Prove that the matrix $I - A$ is invertible.

*Answer.* Suppose that $v \in \mathcal{N}(I - A)$, so $(I - A)v = 0$. Then $v - Av = 0$ or $Av = v$. But now $0 = 0 \cdot v = A^2v = A(Av) = A(v) = Av = v$, so $v = 0$. Thus the null space of $I - A$ is the zero subspace, thus has dimension 0. By the Dimension Theorem, it follows that the rank of $I - A$ is $n$, so $I - A$ is both injective and surjective, hence is an isomorphism; so $I - A$ is invertible.