Recall the Gram-Schmidt orthogonalization algorithm. If \( v_1, v_2, \ldots, v_n \) form a basis of a vector space \( V \) equipped with a specific choice of inner product \( \langle - , - \rangle \), let \( u_1 = \frac{v_1}{\|v_1\|} \).

Then let \( u'_2 = v_2 - \langle v_2, u_1 \rangle u_1 \), and let \( u_2 = \frac{u'_2}{\|u'_2\|} \). One can show this is a nonzero vector orthogonal to \( u_1 \). In general, given \( u_1, \ldots, u_k \), let

\[
    u'_{k+1} = v_{k+1} - \sum_{i=1}^k \langle v_{k+1}, u_i \rangle u_i, \quad u_{k+1} = \frac{u'_{k+1}}{\|u'_{k+1}\|}.
\]

The end result will, I claim, be an orthonormal basis \( u_1, \ldots, u_n \).

**Problem 1.** Apply the Gram-Schmidt algorithm to the following basis of \( \mathbb{R}^3 \) (with the standard inner product):

\[
    v_1 = (1, -1, 0), \quad v_2 = (0, 2, 1), \quad v_3 = (0, 1, 1)
\]

and compute the resulting orthogonal basis \( u_1, u_2, u_3 \).

**Problem 2.** Now use the above basis in a different order,

\[
    v_1 = (0, 1, 1), \quad v_2 = (0, 2, 1), \quad v_3 = (1, -1, 0),
\]

and compute the resulting \( u_1 \) using the Gram-Schmidt method. Does it appear in the orthonormal basis from Problem 1?

**Problem 3.** Define an inner product on the vector space \( \mathbb{R}[x] \) by setting

\[
    \langle p(x), q(x) \rangle = \int_{-1}^{1} p(x)q(x) \, dx.
\]

Applying the Gram-Schmidt algorithm to the basis \( 1, x, x^2, x^3, \ldots \in \mathbb{R}[x] \) gives a sequence of polynomials \( P_0(x), P_1(x), P_2(x), \ldots \) called Legendre polynomials.

1. Compute the Legendre polynomials \( P_0, P_1, P_2, P_3 \).
2. Show that the Legendre polynomial \( P_n \) satisfies the Legendre differential equation (with parameter \( n \)),

\[
    \frac{d}{dx} \left[ (1 - x^2) \frac{dP_n}{dx} \right] + n(n+1)P_n(x) = 0.
\]

Legendre polynomials arise in solving the Laplace equation in spherical coordinates:

http://mathworld.wolfram.com/LaplacesEquationSphericalCoordinates.html

**Problem 4.** Another sequence of orthogonal polynomials are the Hermite polynomials. There we define an inner product on \( \mathbb{R}[x] \) by \( \langle p(x), q(x) \rangle_H = \int_{-\infty}^{\infty} p(x)q(x) \, dx \). The Hermite polynomials \( H_0(x), H_1(x), \ldots \) are not, however, quite obtained by Gram-Schmidt from \( 1, x, x^2, \ldots \): rather they are supposed to satisfy

\[
    \langle H_m(x), H_n(x) \rangle_H = \sqrt{\pi} 2^n n! \delta_{m,n}.
\]

Compute some Hermite polynomials, if you can! You can read lots more here:

https://en.wikipedia.org/wiki/Hermite_polynomials

**Problem 5.** Prove by induction on \( n \) that the Gram-Schmidt method produces an orthonormal basis of \( V \).

Here are a couple of things you might find interesting: applications using inner products on vector spaces:

https://vufind.carli.illinois.edu/vf-uiuc/Record/uiu_7848547