Math 416 Problems

Recall: a transition matrix is an \( n \times n \) matrix \( T \) with all entries \( T_{ij} \geq 0 \), and the sum of entries in the \( j \)th column \( \sum_{i=1}^{n} T_{ij} = 1 \), for every \( j \). A probability vector is a vector \( p = (p_1, \ldots, p_n) \) with each \( p_i \geq 0 \) and \( \sum_{i=1}^{n} p_i = 1 \).

Problem 1. Suppose that \( T \) is a transition matrix. Show that 1 is an eigenvalue of \( T \) by first proving that the characteristic polynomials of \( T \) and its transpose \( T^t \) are the same, and then finding an eigenvector of \( T^t \) with eigenvalue 1.

Problem 2. Consider a game board with 4 boxes, numbered 1, 2, 3, 4 from left to right. A player begins a game of chance by placing a marker in box 2. A die is rolled, and the marker is moved one square to the left if a 1 or a 2 is rolled and one square to the right if a 3, 4, 5, or 6 is rolled. This process continues until the marker lands in square 1, in which case the player wins the game, or in square 4, in which case the player loses the game. What is the probability of winning this game?

Problem 3. In 1940, a county land-use survey showed that 10% of the county land was urban, 50% was unused, and 40% was agricultural. Five years later, a follow-up survey revealed that 70% of the urban land had remained urban, 10% had become unused, and 20% had become agricultural. Likewise, 20% of the unused land had become urban, 60% had remained unused, and 20% had become agricultural. Finally, the 1945 survey showed that 20% of the agricultural land had become unused while 80% remained agricultural. Assuming that the trends indicated by the 1945 survey continue, compute the percentages of urban, unused, and agricultural land in the county in 1950 and the corresponding eventual percentages.

Problem 4. Suppose that \( T \) is a transition matrix and \( x = (x_1, \ldots, x_n) \) is an eigenvector of \( T^t \), so \( T^t x = \lambda x \). Suppose that \( j \) is the index for which the absolute value \( |x_j| \) is largest among \( |x_1|, |x_2|, \ldots, |x_n| \). Show that \( |\lambda||x_j| = |(T^t x)_j| \leq |x_j| \), and thus conclude that \( |\lambda| \leq 1 \). Conclude that every eigenvalue of \( T \) also has absolute value at most 1.