**Math 416 Problems**

**Problem 1.** Let \( A = \begin{pmatrix} 3 & 0 & 3 \\ 2 & 6 & -2 \\ 3 & 0 & 3 \end{pmatrix} \).

1. Compute the characteristic polynomial of \( A \). Compute the eigenvalues of \( A \).
2. For an eigenvalue \( \lambda \) of \( A \), the \( \lambda \)-eigenspace of \( A \) is the null space \( N(A - \lambda I) \). Compute the eigenspace of \( A \) for each eigenvalue of \( A \).
3. Is there a basis of \( \mathbb{R}^3 \) consisting of eigenvectors of \( A \)? If yes, find one, and compute a matrix \( Q^{-1} \) such that \( QAQ^{-1} \) is a diagonal matrix.

**Problem 2.** Suppose a \( 3 \times 3 \) matrix \( B \) has as its characteristic polynomial \( p_B(t) = (-t)(t-6)^2 \). Suppose \( B \) is diagonalizable. Find all possible diagonal matrices \( D \) that are similar to \( B \).

**Problem 3.** Let \( A_t = \begin{pmatrix} 1 & 1 \\ 0 & t \end{pmatrix} \); this is a matrix that depends on the variable \( t \) (if you like you can think of it as a function \( \mathbb{R} \to \text{Mat}_{2 \times 2}(\mathbb{R}) \)). For which values of the variable \( t \in \mathbb{R} \) is \( A_t \) diagonalizable? Justify.

**Problem 4.** Suppose that \( T : V \to V \) is a linear operator, where \( V \) is a finite-dimensional vector space. Suppose that \( W \subseteq V \) is any subspace of \( V \) for which \( T(W) \subseteq W \) (that is, \( T(w) \in W \) for every \( w \in W \)), and write \( T|_W : W \to W \) for the restriction of \( T \) to \( W \). Show that the eigenvalues of \( T|_W \) are a subset of the eigenvalues of \( T \).

**Problem 5.** Let \( N = \begin{pmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & 0 & 0 & \ldots & 0 \end{pmatrix} \) of size \( n \times n \).

1. Compute the characteristic polynomial of \( N \).
2. Show that \( N^n = 0 \) but that \( N^{n-1} \neq 0 \). [The main challenge may be to find a good formula for \( N \).]

**Problem 6.** Suppose that \( V \) has dimension \( n \) and that \( T : V \to V \) is a linear operator whose characteristic polynomial is \( p_T(t) = (-t)^n \). Show by strong induction on \( n \) that \( T^n = 0 \), where \( T^n \) means \( T \circ T \circ \cdots \circ T \), the composition of \( T \) with itself \( n \) times. We say \( T \) is \textit{nilpotent of index at most} \( n \). [Hint: for \( n = 1 \) this is easy. Now for the induction step, assume true for linear operators on \( k \)-dimensional vector spaces where \( k \leq n - 1 \), and consider \( V \) of dimension \( n \). Show that \( \mathcal{R}(T) \) is at most \( (n - 1) \)-dimensional, and that \( T(\mathcal{R}(T)) \subseteq \mathcal{R}(T) \). Apply the inductive hypothesis to \( T|_{\mathcal{R}(T)} \) using the result of Problem 4 etc.]

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