Math 416 Problems

Problem 1. Let \( A = \begin{pmatrix} 3 & 0 & 3 \\ 2 & 6 & -2 \\ 3 & 0 & 3 \end{pmatrix} \).

1. Compute the characteristic polynomial of \( A \). Compute the eigenvalues of \( A \).
2. For an eigenvalue \( \lambda \) of \( A \), the \( \lambda \)-eigenspace of \( A \) is the null space \( N(A - \lambda I) \). Compute the eigenspace of \( A \) for each eigenvalue of \( A \).
3. Is there a basis of \( \mathbb{R}^3 \) consisting of eigenvectors of \( A \)? If yes, find one, and compute a matrix \( Q^{-1} \) such that \( QAQ^{-1} \) is a diagonal matrix.

Solution. The characteristic polynomial is \( p_A(t) = -t(6 - t)^2 \). The eigenvalues are 0 and 6. The 6-eigenspace of \( A \), i.e., the null space of \( A - 6I \), is \( \{(a, b, a) \mid a, b \in \mathbb{R}\} \). The 0-eigenspace of \( A \), i.e., the null space of \( A \), is \( \{(c, -2c/3, -c) \mid c \in \mathbb{R}\} \). One possible basis of eigenvectors is \((1, 0, 1), (0, 1, 0), (1, -2/3, -1)\). We then get

\[
Q^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2/3 \\ 1 & 0 & -1 \end{pmatrix}.
\]

I will leave it to you to compute the inverse, but remember you can do this by forming the \( 3 \times 6 \) matrix \((Q^{-1}|I_3)\) and finding its RREF, which will be \((I_3|Q)\).

Problem 2. Suppose a \( 3 \times 3 \) matrix \( B \) has as its characteristic polynomial \( p_B(t) = (-t)(t-6)^2 \). Suppose \( B \) is diagonalizable. Find all possible diagonal matrices \( D \) that are similar to \( B \).

Solution. A diagonal matrix similar to \( B \) has the same characteristic polynomial as \( B \). If \( D \) is a diagonal matrix with diagonal entries \( d_1, d_2, d_3 \) then its characteristic polynomial is easily computed to be \( p_D(t) = (d_1 - t)(d_2 - t)(d_3 - t) \). It follows that for any diagonal matrix similar to \( B \) above has diagonal entries 0, 6, 6 in some order. Any two diagonal matrices with the same unordered list of diagonal entries are similar! [You can check this yourself in the \( 3 \times 3 \) case.]. Thus, the possibilities are

\[
\begin{pmatrix} 0 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix}, \begin{pmatrix} 6 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 6 & 0 \end{pmatrix}, \begin{pmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

\( \square \)

Problem 3. Let \( A_t = \begin{pmatrix} 1 & 1 \\ 0 & t \end{pmatrix} \); this is a matrix that depends on the variable \( t \) (if you like you can think of it as a function \( \mathbb{R} \to \text{Mat}_{2 \times 2}(\mathbb{R}) \)). For which values of the variable \( t \in \mathbb{R} \) is \( A_t \) diagonalizable? Justify.

Solution. The characteristic polynomial of \( A_t \) is \( p_{A_t}(s) = (1 - s)(t - s) \). This has distinct roots if \( t \neq 1 \), so \( A \) is diagonalizable if \( t \neq 1 \). If \( t = 1 \), then the only eigenvalue of \( A_t \) is 1; in that case, if \( A_1 \) were diagonalizable, we would need to have nullity\((A - I_2) = 2 \), but \( A_t - I_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \), which is not the zero matrix, so its nullity is less than 2 = \( \text{dim}(\mathbb{R}^2) \). Thus if \( t = 1 \) then \( A_t \) is not diagonalizable. \( \square \)

Problem 4. Suppose that \( T : V \to V \) is a linear operator, where \( V \) is a finite-dimensional vector space. Suppose that \( W \subseteq V \) is any subspace of \( V \) for which
We say that an integer \( k \) is \( \leq n \) times. Now for the induction step, assume true for linear operators on a \( k < n \)-dimensional vector space with characteristic polynomial \( p \). If \( T \) is any linear operator on a \( k < n \)-dimensional vector space, so \( T^k \) is nilpotent of index at most \( n \). Thus, by the inductive hypothesis, we have \( p(T) = 0 \), proving the inductive step, hence completing the proof.

**Problem 5.** Let \( N = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \) of size \( n \times n \).

1. Compute the characteristic polynomial of \( N \).
2. Show that \( N^n \) is upper triangular with all diagonal entries equal to \( -t \), so \( p_N(t) = \det(N - tI_n) = (-t)^n \). A good way of describing \( N \) is that, on the standard basis, \( Ne_j = e_{j-1} \) for \( j > 1 \), and \( Ne_1 = 0 \). Then for every standard basis vector, \( N^k e_j = e_{j-k} \), where we interpret \( e_\ell \) to be the zero vector for \( \ell \leq 0 \). So \( N^n e_j = 0 \) for all \( j \), thus \( N^n = 0 \); whereas \( N^{n-1} e_n = e_1 \), nonzero.

**Problem 6.** Suppose that \( V \) has dimension \( n \) and that \( T : V \rightarrow V \) is a linear operator whose characteristic polynomial is \( p_T(t) = (-t)^n \). Show by strong induction on \( n \) that \( T^n = 0 \), where \( T^n \) means \( T \circ T \circ \cdots \circ T \), the composition of \( T \) with itself \( n \) times. We say \( T \) is nilpotent of index at most \( n \). [Hint: for \( n = 1 \) this is easy. Now for the induction step, assume true for linear operators on \( k \)-dimensional vector spaces where \( k < n - 1 \), and consider \( V \) of dimension \( n \). Show that \( \mathcal{R}(T) \) is at most \( (n-1) \)-dimensional, and that \( T(\mathcal{R}(T)) \subseteq \mathcal{R}(T) \). Apply the inductive hypothesis to \( T|_{\mathcal{R}(T)} \) using the result of Problem 4 etc.]

**Solution.** If \( T \) is a linear operator on a \( 1 \)-dimensional vector space \( V \) with characteristic polynomial \(-t\), then the matrix of \( T \) in any basis of \( V \) must be \( 0 \). So \( T^1 = T = 0 \). Now, assume that for all \( k < n \), if \( T \) is any linear operator on a \( k \)-dimensional vector space with \( p_T(t) = (-t)^k \) then \( T^k = 0 \). Let \( V \) be an \( n \)-dimensional vector space and \( T : V \rightarrow V \) a linear operator. Let \( W = \mathcal{R}(T) \). Then for every \( w \in W \), we have \( T(w) \in \mathcal{R}(T) \), i.e. \( T(W) \subseteq W \). Assume also that \( p_T(t) = (-t)^n \); then 0 is an eigenvalue of \( T \), so \( \mathcal{N}(T) \neq \{0\} \), and hence \( \dim(W) < n \). By the result of Problem 4, we have that the eigenvalues of \( T|_W \) are a subset of the eigenvalues of \( T \); but by assumption the only eigenvalue of \( T \) is 0, so the only eigenvalue of \( T|_W \) is zero, and writing \( k = \dim(W) < n \) we have that the characteristic polynomial of \( T|_W \) is \( (-t)^k \). Thus, by the inductive hypothesis, we have \( (T|_W)^k = 0 \). Now for any \( v \in V \), \( T^n(v) = T^{n-1}(w) \) where \( w = T(v) \in W \), and since \( n - 1 \geq k \) we have \( T^{n-1}(w) = 0 \). Thus \( T^n = 0 \), proving the inductive step, hence completing the proof.