last time used lift to $M_g(n)$ of conical action on $U(1)^n/G$, then adjusted by a Hamiltonian action, to try to compute Betti numbers of $M_g(n)$ using Biatyukii-Birkula. Not clear why it would always work, though (and in fact it needn’t—well, in a sense that fixed point sets may be complicated).

Instead I will explain a different approach, carried out by me in joint work w/ McCarthy.

**Idea:** $M$ a smooth variety. Suppose we could describe the Poincaré dual to $[\Delta]$, $\Delta \subset M \times M$ diagonal, as $[\Delta] = \sum \text{Pic} \cdot \text{up}_{\mathbb{P}^2 di}(\chi) \cdot M \times M \overset{\text{proj}_{\mathbb{P}^2}}{\longrightarrow} M$.

Would like to use projection formula:

$$p^*_x([\Delta] \cup p^*_2 \alpha) = p^*_x([x \cdot p^*_2 \alpha], \quad \Delta \overset{i}{\longrightarrow} M \times M$$

$$= i^* p^*_2 \alpha = \alpha.$$

Then replace $[\Delta]$ by RHS of ($\chi$), get

$$\alpha = p^*_x(\sum p^*_x \text{Pic} \cdot \text{up}_{\mathbb{P}^2 di} \cup p^*_2 \chi) = p^*_x(\sum p^*_x \text{Pic} \cup p^*_2 \chi (\text{div} \chi))$$

$$= \sum\text{Pic} \cdot (\sum_{\text{div} \chi}).$$

So, we would see that

$$\text{Pic} \text{ span } H^*(M).$$

There are issues with the above “argument,” however.
(1) we repeatedly use integration over $M$ — every time we want to apply $p_!$, it's integration over fibers. That won't work unless $M$ is compact.

**Analogously** on sheaves, want to use

$$p_!(\mathcal{O}_\Delta \otimes p_2^* \alpha) = p_!(i_* i^! \mathcal{O}_M \otimes p_2^! \alpha)$$

$$= p_!(i_* i^! p_2^* \alpha) = \alpha'$$

replace $\mathcal{O}_\Delta$ by resolution

$$0 \to \mathcal{E}_r \to \ldots \to \mathcal{E}_0 \to \mathcal{E}_0 \to \mathcal{O} \to 0 \quad (\ast)$$

get $\alpha \equiv (\mathcal{E}_r \otimes H^0(\mathcal{F} \otimes \alpha)) \to \ldots \to \mathcal{E}_0 \otimes H^0(\mathcal{F}_0 \otimes \alpha))$

**But** if $M$ is noncompact, $H^0(\mathcal{F}_i \otimes \alpha)$ is probably infinite-dimensional. Hard to use to get liftof a coherent sheaf $\alpha$.

(2) Anyway, how would you ever find $(k)$, or $(k^n)$?

It turns out both problems are best solved by **viewing** for quiver varieties by viewing $\nu \mathcal{M}(k)$ as "moduli space of stable sheaves on Spec $T_0^n$" as an inspiration for compactifying $\mathcal{M}(k)$. 
Example: Recall the case that gave \( M(n) \cong (\mathbb{P}^1)^n \),

i.e., \( Q = \mathbb{C}^{n \times n}, \quad G = \text{GL}(n), \quad x(g) = \det(g) \).

Here the preprojective algebra is a quotient of

\[ \mathbb{C}Q^{dbl} = \mathbb{T}_0^{\circ} \langle x \otimes e_1 \otimes e_2 \otimes \ldots \rangle, \]

\[ e_1^2 = 1, \quad e_1 e_2 = e_2 e_1, \quad e_1 x = 0 = x e_1, \quad e_2 y = 0 = y e_2, \]

Also have \( p = (xy - yx + ij) \) at vertex 2.

So \( \mathbb{C}Q^{dbl} / p \cong \mathbb{T}_0^{\circ} \) doesn't really look like \( \mathbb{C} \langle xy \rangle \),

but like some modified version. Now the loss there's

some open set of modules or "sheaves on \( \text{Spec } \mathbb{T}_0^{\circ} \)"

i.e. \( \mathbb{T}_0^{\circ} \)-modules, that looks like an open set of

modules of sheaves on \( \text{Spec } \mathbb{C} \langle xy \rangle \).

Could try to use either to compactify.

\( \mathbb{A}^2 \setminus \{0\} \mapsto (\mathbb{P}^1)^2 \setminus \{0\}, \) Hilbert scheme of points on \( \mathbb{P}^2 \).

It's smooth and projective!

But I have no real idea how to generalize.

Take as inspiration! Find "Proj \( A \)" with

a localization \( A[T]/T \cong \mathbb{T}_0^{\circ} \).

Simple answer: \( A = \mathbb{T}_0^{\circ}[T], \) graded so that

\( \deg(a) = 1 \) for \( a \in H, \quad \deg(e_i) = 0, \quad \deg(t) = 1. \)
Aside. This is almost always a $3$-Calabi-Yau algebra, Jacobian algebra of triplet quiver $Q$ obtained by adding a loop at each vertex, labelled $t_v$, and using superpotential $\sum \left( \sum_{i \in I} E(a) a^a t_i \right)_{s(a) = i}$.

I would like to compactify $M_g(n)$, then, by taking a moduli space of $\text{"sheaves on Proj } A\text{"}$, i.e. graded $A$-modules modulo equivalence relation.

Recall

Thm [Serre] If $R$ is a $k$-g. comm. $k$-algebra, nonnegatively graded with $R_0 = k$, and generated in degree $1$, then

$$\text{Col(Proj } A) \cong A\text{-mod}/\text{Tails } (A),$$

where $A\text{-mod}$ is category of $k$-g. graded $A$-modules,

$\text{Tails } (A) = \text{category of bounded graded } A\text{-modules}$,

i.e. $M$ s.t. $M_k = 0 \forall k \gg 0$.

[Recall constructions of $M \rightarrow \hat{M}$.

$T(\chi) < T$.]
Now suppose that $A$ is any nonnegatively graded (w/ $A_0$ semisimple, say) associative algebra. Consider

$$\text{Coh (Proj} A) = A \text{mod } \text{Tor}(A),$$

even without making sense of any space $\text{Proj} (A)$. The goal is to "understand"/describe a moduli's space of objects in $\text{Coh (Proj} A)$.

Suppose $\mathcal{F}$ is a sheaf on a proj. variety $Y$ with $0$-dim'l support. Then $\mathcal{F} \otimes \mathcal{O}(n) \cong \mathcal{F}$, non-canonically, and thus $\mathcal{F} \otimes \mathcal{O}(n) \cong \mathcal{F}$ for all $n \geq 0$. Thus

\[ \mathbb{P}^n \mathcal{F} \text{ has constant Hilbert series:} \]

\[ \text{dim } \mathbb{P}^n \mathcal{F} = \text{dim } H^0(Y, \mathcal{F} \otimes \mathcal{O}(n)) = \text{dim } H^0(Y, \mathcal{F}) \]

for all $n \geq 0$.

In some sense, we have $\mathcal{M} = \mathbb{P}^n \mathcal{M}$, so to study $0$-dim'l sheaves (i.e. sheaves with $0$-dim'l support) on $Y = \text{Proj}(R)$ we may start with $\mathcal{M}$ as an $A$-module of constant Hilbert series.

Fact in projective algebraic geometry. For any bounded family of coherent sheaves $\{ \mathcal{F}_x \}$, there is an $A(\alpha)$ sit.

\[ \mathbb{P}^n (\mathcal{F}_x) \to A(\alpha) \]

as a graded $R$-module, determines $\mathcal{F}_x$. 
Naive Hope: If $R$ is generated in deg 1 and is quadratic, then $\Gamma^*(g)/\Gamma^{\geq 3}(g) \to \infty$ is enough if $g$ has 0-dimensional support.

Use this as an ansatz to prescribe a compactification of $\text{M}_0(n)$.

Graded-tripled quiver: Given quiver $Q$, vertex set $I$, edge set $\Omega$, $Q^{gr}tr$ has

vertices $I \times \{0,1,2\}$,

edges $\Omega \times \{0,1\}$ $\alpha$, $\overline{\alpha} \times \{0,1\}$ $\alpha$, $I \times \{0,1\}$ $\overline{\alpha}i,j$.

$s(\alpha) = (\alpha, \{0\}, s, t(\overline{\alpha}) = (t(\alpha), 1)$
$s(\overline{\alpha}) = (t(\alpha), \{0\}, s, t(\alpha) = (s(\alpha), \{\})$
$s(\overline{\alpha}) = (i, j)$ $t(\overline{\alpha}) = (i, j+1)$.

Ex. \[ \begin{array}{ccc}
\alpha & \rightarrow & \overline{\alpha} \\
\alpha & \leftarrow & \overline{\alpha}
\end{array} \]
Recall \( A = T^0[\tau] \), graded by \( e_i \in A_0 \), \( a \in A_1 \), \( \omega \in A_2 \), \( t \in A_1 \).

Suppose \( B \) is a graded \( C \)-algebra. Define
\[
B[s] = \{ \sum_i b_i s^i \mid b_i \in B_i \}, \\
\text{with } B = B_0 s^0 < B[s],
\]
as a subalgebra and \( s - b = b(s + k) \), \( b \in B_k \).

i.e. \([s, b] = kb, b \in B_k\).

Given graded left \( B \)-module \( M \), get \( B[s] \)-module structure on \( M \)
by letting \( s \cdot m = km \), \( m \in M_k \), acts semisimply on \( M \),
i.e. \( M = \bigoplus_{\lambda} M_{\lambda} \) where \( s \cdot m = \lambda m \) \( \forall \lambda \in \mathbb{C} \).

Conversely, given \( s \)-semisimple \( B[s] \)-module \( M \)
with \( s \)-eigenvalues in \( \mathbb{Z} \), grade \( M \) by \( s \)-eigenvalue
and get a graded \( B \)-module.

**Lemma** The functors
\[
\begin{array}{ccc}
B-Gr & \xrightarrow{\cong} & B[s]-\text{Mod}_{\mathbb{Z}-ss} \\
\text{graded left} & \text{B-modules} & \text{left B[s]-modules on which}
\end{array}
\]
acts semisimply with
integral eigenvalues
are equivalences of categories.

Similarly, can form $A[s^3] = (A/A_{\geq 3})^s[s^3]$, as well as $TT^0Q[t] \{ s \} = (TT^0Q[t] / TT^0Q[t]_{\geq 3}) \{ s \}$.

We define a map

$$CQ^{gr} \{ s^3 \} \rightarrow TT^0Q[t] \{ s \} \text{ by }$$

\[
\begin{align*}
s & \mapsto s \\
\text{edge} \rightarrow e_{ij} & \mapsto te_i e_s(s-1)(s-2) \frac{s-j}{s-j} & j = 0, 1 \\
\text{edge} \rightarrow a_j & \mapsto a_s(s-1)(s-2) \frac{s-j}{s-j} & j = 0, 1 \\
\text{non-edge} \rightarrow e_{ij} & \mapsto e_i e_s(s-1)(s-2) \frac{s-j}{s-j} & j = 0, 1, 2
\end{align*}
\]

The induced functor $TT^0Q[t] - \text{Mod} \rightarrow CQ^{gr} - \text{Mod}$ is an equivalence of categories. The projective relation $q^{[t]}$ maps to $\sum q^{[t]} q_0$, relation that $t$ is central.

The induced relation $q^{[t]}$ maps to $q e_s a_0 = e_s a_0 q_0$ for $a_0 e_s H$.

Thus, we can use $CQ^{gr} \{ s \} / \{ t \}$ to try to construct compactification of $M_g$.
So, consider

\[ \text{Rep}(\mathbb{Q}^{gtr}, n) \supset \text{Rep}(\bar{\mathbb{A}}, n) = \{ M \mid \text{rels (1) satisfied in } M \} \]

steps \( M \) with

\[ \dim \ e_{i;j} M = N_i \]

\( j = 0,1,2 \)

Have group \( G^{gtr} = G \times \mathbb{R} \times \mathbb{R} \) acting, with \( G \times \mathbb{R} \times \mathbb{R} \) acting on \( j \)-th layer of vertices.

Prop For any character \( \Theta : G^{gtr} \to \mathbb{G}_m \), the GIT quotients

\[ \text{Rep}(\mathbb{Q}^{gtr}, n) / G^{gtr}, \Xi \]

\[ \text{Rep}(\bar{\mathbb{A}}, n) / G^{gtr} \Xi \]

are either empty or projective.

Reason No nonconstant functions - quivers have no cycles.

Now, suppose \( M \) is a \( C^{gtr} \)-module with all arrows

\[ e_{i;j} : M_j \to M_{j+1} \]

acting invertibly, i.e., as isomorphisms.

Use them to define isomorphisms

\[ M_j = \bigoplus_{i \in I} e_{i;j} M \to M_{j+1} \quad \forall j = 0,1, \]

Use these to define \( T^{0} \)-action by:

\[ a \mapsto a^{-1} \]

\[ e_{i;j} \mapsto e_{i;j}^{-1} e_{i,0} e_{i,0} \]

Check It does satisfy relations.

Get functors from \( \bar{\mathbb{A}} \)-modules w/ \( e_{i;j} \)'s invertible to \( T^{0} \)-modules.