Recall

**Def.** Subset \( S \subseteq V \) is linearly dependent if there exist pairwise distinct vectors \( v_1, \ldots, v_k \in S \) for some \( k \geq 1 \), and scalars \( c_1, \ldots, c_k \), not all zero, such that

\[
0 = c_1v_1 + c_2v_2 + \ldots + c_kv_k
\]

Otherwise, \( S \) is linearly independent.

**How to show linearly dependent?** Find some vectors \( v_1, \ldots, v_k \) and scalars \( c_1, \ldots, c_k \).

**How to show linearly independent?** Prove that for every \( \{v_1, \ldots, v_k\} \subseteq S \), pairwise distinct, the only solution of

\[
c_1v_1 + \ldots + c_kv_k = 0
\]

is \( c_1 = c_2 = \ldots = c_k = 0 \).

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**Rank:** If \( S = \{v_1, \ldots, v_k\} \), pairwise distinct, enough to check for \( c_1v_1 + \ldots + c_kv_k = 0 \) for \( k \). (No need to study proper subsets of \( S \).)

**Def.** A subset \( S \subseteq V \) generates \( V \) if span(\( S \)) = \( V \).

A subset \( S \) is a basis for \( V \) if \( S \) is linearly independent and span(\( S \)) = \( V \).

**Example:** \( V = \mathbb{R}[x] \), the vector space of polynomials \( p(x) \) with real coefficients. Elements are expressions of the form \( p(x) = a_0 + a_1x + \ldots + a_dx^d \), where \( a_0, \ldots, a_d \in \mathbb{R} \). How to add? Multiply by a scalar?
Lemma. The function \( R[x] \xrightarrow{F} \text{Fun}(R,R) \)
that takes a formal expression \( p(x) = a_0 + a_1 x + \ldots + a_n x^n \)
to the function \( F(p) \) defined by \( F(p)(x) = p(x) \in R \), is one-to-one.

Proof. If \( p(x) = a_0 + a_1 x + \ldots + a_n x^n \), \( q(x) = b_0 + b_1 x + b_2 x^2 \), and \( F(p)(x) = F(q)(x) \) for all \( x \in R \), then
\[
F(p-q)(x) = F(p)(x) - F(q)(x) = 0 \text{ for all } x \in R.
\]
Then the polynomial \( p(x) - q(x) \) has as its roots all real numbers. But, for any polynomial \( r(x) \), if \( r(c) = 0 \), then one can write \( r(x) = (x-c)s(t) \) for a polynomial \( s(t) \) with \( \deg(s) = \deg(r) - 1 \). So if \( r(x) \) is nonzero, it can have at most \( \deg(r) \) distinct roots. Thus, \( p(x) - q(x) \) is the zero polynomial, i.e. \( p(x) = q(x) \) as polynomials, as desired. \( \square \)

Claim. \( \{1, x, x^2, x^3, \ldots\} \) is a basis of \( R[x] \).
- It spans: every polynomial is a linear combination of the monomials.
- It is linearly independent: if for some \( c_i \geq 0 \),
\[
c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n = 0 \text{ as polynomials},
\]
then \( c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n = 0 \) as functions, contradicting \( \square \)
then by definition of polynomials \( c_0 = c_1 = \ldots = c_n = 0 \),
so \( 1, x, x^2, \ldots \) are linearly independent. \( \square \)
Easier: Let \( e_i \) in \( \mathbb{R}^n \) be vector \( e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \) with \( i \)th place.

Then \( e_1, \ldots, e_n \) is a basis for \( \mathbb{R}^n \), standard basis.

Pf of both parts:

\[
Ce_1 + \ldots + C_n e_n = (C_1, C_2, \ldots, C_n).
\]

Each vector in \( \mathbb{R}^n \) has a unique expression as a linear combination of \( e_1, \ldots, e_n \). [Why is that enough?]

Theorem 1: If a vector space \( V \) has a finite spanning set \( S \) then \( V \) has a finite basis.

Pf: Last time:

\( \text{Thm 2: If finite set } S \subseteq V \text{ is non-empty, then there exists a linearly independent subset } \{v_1, \ldots, v_k \} \subseteq S \text{ with } \text{span} \{v_1, \ldots, v_k\} = \text{span} S \).

Pf of Thm 2: If \( S = \{v_1, \ldots, v_r\} \) and it's not lin. indep., then \( \exists \ c_1, \ldots, c_r \text{ not all zero s.t. } c_1 v_1 + \ldots + c_r v_r = 0. \)

Re-order so \( c_1 \neq 0 \). Then

\[
v_1 = -\frac{c_2}{c_1} v_2 - \ldots - \frac{c_r}{c_1} v_r.
\]

Now 
\[
d_1 v_1 + \ldots + d_r v_r = d_1 \left( -\frac{c_2}{c_1} v_2 - \ldots - \frac{c_r}{c_1} v_r \right) + d_2 v_2 + \ldots + d_r v_r = \left( -\frac{d_1}{c_1} c_2 + d_2 \right) v_2 + \ldots + \left( -\frac{d_1}{c_1} c_r + d_r \right) v_r, \text{ so }
\]

\( \text{span} \{v_1, \ldots, v_r\} = \text{span} \{v_2, \ldots, v_r\} \). Thus we can shrink \( S \) by one element and keep same span. Repeat until only one element or set linearly independent, whichever comes first.
**Theorem** A subset \( \{v_1, \ldots, v_n\} \subseteq V \) is a basis if and only if every \( v \in V \) can be uniquely expressed as a linear combination \( v = c_1 v_1 + \cdots + c_n v_n \) of \( v_1, \ldots, v_n \).

**Thm** (For next time) For every vector space with a finite basis, the number of elements in every basis is the same.

**Def** If \( V \) has a finite basis \( \{v_1, \ldots, v_n\} \), we say the **dimension** of \( V \) is \( n \). In particular, \( V \) is **finite-dimensional**. If \( V \) has no finite basis, \( V \) is **infinite-dimensional**.

**Ex** \( \mathbb{R}^k \) is infinite-dimensional. [Almost implied by above.]

**Thm** Suppose \( W \subseteq V \) is a subspace, \( \dim V = n < \infty \).

Then \( \dim W \leq \dim V \), and \( \dim W = \dim V \) if \( W = V \).
Last Time: No vector space, \( S \subseteq V \) a subset.
- \( S \) spans or generates \( V \) if \( \text{span}(S) = V \).
- \( S \) is a basis for \( V \) if (i) \( S \) is linearly independent, (ii) \( S \) spans \( V \).

Replacement Theorem: Suppose \( V \) is a vector space spanned by a set \( S \) with \( n \) vectors, and \( L \) is a linearly independent set in \( V \) with \( k \) vectors. Then \( k \leq n \) and there exists a subset \( T \subseteq S \) with \( n-k \) vectors so that \( LUT \) spans \( V \).

We'll come back to this in a minute. First:

Corollary: If the vector space \( V \) has some finite basis, then any two bases have the same number of elements.

Proof: If \( S, S' \) are bases, \( |S| \neq |S'| \), then choose a subset \( L \subseteq S \) with \( k+1 \) elements.

By Replacement Theorem using \( L \) lin. indp. and \( S' \) spanning, get \( k+1 \leq k \), contradiction. So two bases cannot have different numbers of elements. \( \square \)

Proof of Replacement Theorem: By induction on \( k \).

Remember Principle of Induction: If there is a statement \( P_k \) about an integer \( k \geq 0 \), and if
- (1) \( P_0 \) is true, and
- (2) for every \( k \), if \( P_k \) is true then \( P_{k+1} \) is true,
then \( P_k \) is true for all \( k \geq 0 \).
Case \( k = 0 \): Then \( L \) is empty set, \( T = S \) works. So assertion true for \( k = 0 \).

**Induction Step:** Assume assertion is true for set \( L \) with \( k \) vectors. Suppose \( L \) has \( k+1 \) vectors, \( v_1, v_2, \ldots, v_{k+1} \), and let \( L' = \{v_1, v_2, \ldots, v_k\} \). By inductive hypothesis, there is subset \( T' = \{w_1, \ldots, w_{n-k}\} \) of \( S \)

so \( L' \cup T' = \{v_1, v_2, \ldots, v_k, w_1, \ldots, w_{n-k}\} \) spans \( V \).

Use that \( \text{span} \) to write

\[
(1) \quad v_{k+1} = c_1 v_1 + \cdots + c_k v_k + d_1 w_1 + \cdots + d_{n-k} w_{n-k}
\]

Note that some of \( d_1, \ldots, d_{n-k} \) must be nonzero, otherwise \((1)\) says \( v_1, v_2, \ldots, v_k \) are linearly dependent, contradiction. Re-ordering \( w_i \)'s, we may assume \( d_j \neq 0 \).

Then \( w_1 = -\frac{1}{d_1}(-v_{k+1} + c_1 v_1 + \cdots + c_k v_k + d_2 w_2 + \cdots + d_{n-k} w_{n-k}) \)

so \( w_1 \in \text{span} \{v_1, v_2, \ldots, v_k, w_2, \ldots, w_{n-k}\} \). Thus

\[
\text{span}(L \cup \{w_2, \ldots, w_{n-k}\}) = \text{span}(L \cup T')
\]

\[
= \text{span}(L' \cup T') = V, \quad \text{Thus taking } T = \{w_2, \ldots, w_{n-k}\}
\]

gives \( \text{span}(L \cup T) \), proving the induction step, hence the theorem.

**Cor.** Any linearly independent subset of a vector space has size \( \leq \) the size of a basis. Every linearly independent subset can be extended to a basis.
A system of linear equations, if homogeneous and all constants are 0:
\[ a_1 x_1 + \ldots + a_k x_k = 0 \]
\[ a_2 x_1 + \ldots + a_k x_k = 0 \]
\[ a_m x_1 + \ldots + a_k x_k = 0 \]
Then set of solutions \((x_1, \ldots, x_k)\) is the null space, \(\text{Null}(A)\), of coefficient matrix \(A\).

\[ \text{null}(A) \]

How do you find a basis for \(\text{Null}(A)\)? How do you compute its dimension?

Recall: Use Gauss-Jordan elimination to convert \(A\) to a matrix \(B\) in RREF.

\[ B = \begin{pmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \]

equations:
\[ x_1 + 3x_2 + x_4 = 0 \]
\[ x_3 - x_4 = 0 \]
Free vars: \(x_2, x_4\).

Parameterized solutions:
\[ \{ (-3u-v, u, v, v) \mid u, v \in \mathbb{R} \} = \{ u(-3, 1, 0, 0) + v(-1, 1, 1) \mid u, v \in \mathbb{R} \} \]

Claim: These are linearly independent.

Proof: \(u, v\) are coeff of the linear comb, so if linear comb.
\[ 0 \]
\[ u = 0 = v \]

Then: Suppose \(A \in \text{Matrix}(\mathbb{R})\), \(B\) is a row-equivalent matrix in RREF, with \(p\) pivot columns (so \(n-p\) free variables).

Then \(\dim \text{Null}(A) = n-p\).
Let \( u_1, \ldots, u_{n-p} \) be solutions of \( LS(B|0) \) where \( u_i \) has \( i \)-th free variable \( = 1 \), other free variables \( = 0 \).

Then these vectors \( u_1, \ldots, u_{n-p} \) form a linearly independent set.

If \( c_1 u_1 + \cdots + c_{n-p} u_{n-p} = 0 \), then the entry of vector \( c_1 u_1 + \cdots + c_{n-p} u_{n-p} \) corresponding to the free variable \( i \) is \( c_i \) but also is zero, so \( c_i = 0 \) for all \( i \).

Now any solution is determined by its values of the free variables, and \( c_1 u_1 + \cdots + c_{n-p} u_{n-p} \) is a solution with \( i \)-th free variable taking the value \( c_i \), so \( u_1, \ldots, u_{n-p} \) span \( N(B) = N(A) \).

**Def.** The row space \( R(A) \) of \( A \in \text{Mat}_{m \times n}(\mathbb{R}) \) is the span of its rows, understood as elements of \( \mathbb{R}^n \).

**Ex.** \( \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \), row space \( R(A) = \text{span} \{ (1,2), (3,4), (5,6) \} \).

**Thm.** If \( A \) and \( B \) are row equivalent matrices, then \( R(A) = R(B) \).

**Discuss.**

**Thm.** If \( A \in \text{Mat}_{m \times n}(\mathbb{R}) \) is in RREF, then the nonzero rows of \( A \) form a basis of \( R(A) \).

Use to compute a basis for span:

**Ex.** span \( \{ (1,2,3), (4,5,6), (7,8,9) \} \).

\( \Rightarrow \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \) etc.