A ∈ Matrix (F), or V a finite-dimensional vector space, T: V → V a linear operator, over a field F, perhaps R, perhaps not.

Def For λ ∈ F, the λ-eigenspace of T is
\[ E_λ = \ker(T - λI). \]

Generalized λ-eigenspace is
\[ K_λ = \bigcup_{p=1}^\infty \ker((T - λI)^p). \]

Ex A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, T = L_A, F = R.
\[ K_3 = E_3 = \text{span} \{ e_1 \}. \]

Ex A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, T = L_A, F = R.
\[ E_0 = \text{span} \{ e_1 \}, K_0 = R^2. \] (since \( A^2 = 0 \)).

Ex A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, T = L_A.
\[ A(x, y) = (y, -x), \text{ so if } F = R \text{ get } \]
\[ E_\lambda = \{(x, y) | \lambda(y) = -\lambda x \} = \{(x, y) | \lambda x = y \} = \{(x, y) | x y = -y \} = \{(0, 0)\}. \forall \lambda \in \mathbb{R}. \]

But if F = C, \[ E_i = \text{span} \{ i \} \] and \[ E_{-i} = \text{span} \{ -i \}. \]
Def: A polynomial \( p(x) \in \mathbb{F}[x] \) splits over \( \mathbb{F} \) if
\[
p(x) = c(x - d_1) \cdots (x - d_n)
\]
for some \( c, d_1, \ldots, d_n \in \mathbb{F} \).

Ex: \( (x^2 - 1) = (x+1)(x-1) \) splits over \( \mathbb{R} \),
\( x^2 + 1 \) does not! \( x^2 + 1 = (x-i)(x+i) \) in \( \mathbb{C}[x] \).

**Fundamental Theorem of Algebra**

Every \( p(x) \in \mathbb{C}[x] \) splits over \( \mathbb{C} \).

**Proof:** If \( V \) is a finite-dimensional vector space over \( \mathbb{R} \),
\( T: V \to V \) a linear operator, and \( \mathbf{e} \) is diagonalizable, then \( p_T(\mathbf{e}) \) splits over \( \mathbb{R} \).

**Pt.** \( p_T(\mathbf{e}) = p(\mathbf{T})p(\mathbf{e}) \) for any basis \( \beta \) of \( V \); if \( [\mathbf{T}]_\beta \)
is diagonal, \( [\mathbf{T}]_\beta = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{pmatrix} \), then
\[
p_T(\mathbf{e}) = (d_1 - \mathbf{t}) \cdots (d_n - \mathbf{t}), \quad d_1, \ldots, d_n \in \mathbb{R}.
\]

Indeed, we see from the above that:

**Proof:** If \( V \) is finite-dimensional, \( T: V \to V \) diagonalizable, then \( \dim \mathbb{E}_\lambda = m_\lambda \) where
\[
p_T(\mathbf{t}) = (\mathbf{t} - \lambda)^m \quad \text{for each } \lambda \in \mathbb{R}.
\]

Conversely

**Theorem:** Suppose \( V \) is finite-dimensional over \( \mathbb{R} \), \( T: V \to V \) a linear operator,
and \( p_T(t) = (\lambda_1 - t)^{m_1} \cdots (\lambda_s - t)^{m_s} \) where \( \lambda_1, \ldots, \lambda_s \) are distinct. If \( \dim \mathbb{E}_{\lambda_i} = m_i \), \( \forall i = 1, \ldots, s \), then
\( T \) is diagonalizable.
Choose basis $\{v_i, g_i\}$, $i = 1, \ldots, s$ of each $E_i$.

Then we claim $\{v_i, g_i\}$, $i = 1, \ldots, s$, is a basis of $V$. Enough to show they are linearly independent. [Why?]

Suppose $\sum c_i v_i + \sum d_i g_i = 0$. Collect terms in each $E_i$, so get $w_1 + \cdots + w_s = 0$.

Prop: Each $w_i$, $1 \leq i \leq s$, is zero.

Indeed, we can show by induction:

Base: Suppose $w_1, w_2, w_3$ are any three elements of $E_1, E_2, E_3$, where $\lambda_i \neq 0$ for $i \neq j$.

If $w_1 + w_2 + w_3 = 0$ then all $w_i$ are zero.

Pf: By induction on $s$. $s = 1$ is clear.

If true for $s-1$, consider $w_1 + w_s = 0$.

Then $0 = T(0) = T(w_1 + \cdots + w_s) = T(w_1) + \cdots + T(w_s) = \lambda_1 w_1 + \cdots + \lambda_s w_s$.

At least one $\lambda_i$, say $\lambda_s$, is nonzero. Then

$0 = \frac{\lambda_1}{\lambda_s} w_1 + \cdots + (\frac{\lambda_s-1}{\lambda_s}) w_{s-1} + w_s = 0 \neq (\frac{\lambda_i}{\lambda_s}-1) w_i + \cdots + (\frac{\lambda_s-1}{\lambda_s}) w_{s-1}$.

Now each $(\frac{\lambda_i}{\lambda_s}-1) \neq 0$, $i \neq s$, so by induction hypothesis, $w_1, \ldots, w_{s-1}$ are zero. Hence $w_1 + w_s = 0$ implies $w_s = 0$.

Back to theorem. Now each linear comb of basis elements of a given $E_i$ is zero, so all coefficients $c_i, d_i$ are zero. Thus $\{v_i, g_i\}$ is a basis, as desired. \[\square\]

Cor: If $p(t)$ has $n = \deg(p)$ distinct roots, then $T$ is diagonalizable.
This \( T : V \to V \) a linear operator on a finite-dimensional vector space, \( p_T(t) = (\lambda_1 - t)^{m_1} \cdots (\lambda_r - t)^{m_r} \) with \( \lambda_1, \ldots, \lambda_r \) distinct. If
\[
\dim E_i = \dim N(T - \lambda_i I) = m_i, \quad i = 1, \ldots, r,
\]
then \( T \) is diagonalizable.

We reduced it to:

Proof: If \( \lambda_1, \ldots, \lambda_r \) are distinct eigenvalues of \( T : V \to V \)
and \( w_i \in E_{\lambda_i}, \ i = 1, \ldots, r, \) and
\[
w_1 + \ldots + w_r = 0
\]
then each \( w_i = 0. \)

Proof. By induction on \( r. \quad r = 1: \) clear. Now, suppose \( r \geq 2, \)
and suppose true for \( r-1 \) eigenvalues/eigenvectors, and
that \( \lambda_1, \ldots, \lambda_r \) are distinct eigenvalues of \( T, \ w_i \in E_{\lambda_i}, \)
\[
w_1 + \ldots + w_r = 0.
\]
Then
\[
T(0) = T(0) = T(w_1 + \ldots + w_r) = \lambda_1 w_1 + \ldots + \lambda_r w_r.
\]
Since \( r \geq 2 \) and \( \lambda_1, \ldots, \lambda_r \) distinct, at least one \( \lambda_i \) is nonzero, say \( \lambda_1 \), without loss of generality \( \lambda_r \). Then
\[
0 = \frac{\lambda_1}{\lambda_r} w_1 + \ldots + \frac{\lambda_{r-1}}{\lambda_r} w_{r-1} + w_r \quad \text{and}
\]
\[
0 = w_1 + \ldots + w_{r-1} + w_r; \quad \text{taking difference, get}
\]
\[
0 = (1 - \frac{\lambda_1}{\lambda_r}) w_1 + (1 - \frac{\lambda_2}{\lambda_r}) w_2 + \ldots + (1 - \frac{\lambda_{r-1}}{\lambda_r}) w_{r-1}.
\]
Since \( \lambda_i / \lambda_r \neq 1 \) for \( i = 1, \ldots, r-1, \) the inductive hypothesis implies \( (1 - \frac{\lambda_i}{\lambda_r}) w_i = 0 \) for \( i = 1, \ldots, r-1 \)
implies \( w_i \geq 0 \) for \( i = 1, \ldots, r-1. \) Now \( w_1 + \ldots + w_r = 0 \)
implies \( w_r = 0 \) as well. This proves inductive step, hence the proposition.
Markov Chains

Suppose a student is wandering a single block looking for... something. To simplify the calculations, let's imagine it's a triangular block:

\[
\begin{array}{c}
\text{3} \\
\text{2} \\
\text{1} \\
\end{array}
\]

At each moment, the student is at some corner. A minute later, she has either:

1. stayed put, probability 20%.
2. moved to corner to the left, probability 40%.
3. moved to corner to the right, probability 40%.

Express probability that a student is currently at each of the 3 corners as a vector in \( \mathbb{R}^3 \):
\[
\mathbf{p}^t = (p_1, p_2, p_3), \quad \text{each of } p_1, p_2, p_3 \geq 0,
\]
\[
p_1 + p_2 + p_3 = 1.
\]

Then get the new probabilities a minute later by multiplying \( \mathbf{p}^t \) by a matrix of transition probabilities, or transition matrix:

\[
\mathbf{T} = \begin{pmatrix}
.2 & .4 & .4 \\
.4 & .2 & .4 \\
.4 & .4 & .2 \\
\end{pmatrix}
\]

Properties:

1. All entries of \( \mathbf{T} \) are \( \geq 0 \).
2. Each column sums to 1.
Can we diagonalize $T$, so we can easily see for a given $p$, how to get to “next $p$” i.e. $T(p)$?

$p_T(t) = \det \left( \begin{array}{ccc} 2-t & 4 & 4 \\ 4 & 2t+9 & 4 \\ 4 & 4 & 2-t \end{array} \right) = (1-t)(2t+6)^2.$

$$= (2-t) \left| \begin{array}{ccc} 4 & 4 & 4 \\ 4 & 2t+9 & 4 \\ 4 & 4 & 2-t \end{array} \right| + 4 \left| \begin{array}{ccc} 2-t & 4 & 4 \\ 4 & 2t+9 & 4 \\ 4 & 4 & 2-t \end{array} \right|$$

$$= (2-t)(0.04-4t+9t^2-.16) - 4(0.08-4t-.16)+4(1.16-.08+.96)$$

$$= 0.08-0.04t-0.08t+4t^2+2t^2-t^3-.032+.16t$$

$$= 0.04+3.6t+.6t^2-t^3$$

$E(-.2) = N \left( \begin{array}{ccc} 4 & 4 & 4 \\ 4 & 4 & 4 \end{array} \right) = \{ (x,y,-x-y) \mid x,y \in \mathbb{R} \}$

$$= \text{span} \{ (1,0,-1), (0,1,-1) \}.$$}

$E_1 = N \left( \begin{array}{ccc} -0.8 & 4 & 4 \\ 0.4 & -0.8 & 4 \end{array} \right) = \text{span} \{ (1,1,1) \}.$

Write $\vec{p} = a(1,1,1) + b(1,0,-1) + c(0,1,-1).$

Then after $N$ minutes,

$T^N \vec{p} = T^N (a(1,1,1) + b(1,0,-1) + c(0,1,-1))$

$$= a(1,1,1) + (-2)^N b(1,0,-1) + (-2)^N c(0,1,-1).$$

So for $N$ very large, $T^N \vec{p} \to a(1,1,1).$

$\vec{p}$ a probability vector, so $p_1+p_2+p_3=1$, i.e. $a+b+c=1$. Conclusion: long-term probability is $\left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right)$. 