Math 416 Midterm 2    Name_________________________

Instructions:
(1) Show work/justify answers to all problems unless you are told otherwise.
(2) Electronics must remain OFF and put away throughout the exam.
(3) No books, notes, formula sheets, etc. are allowed.
(4) You must stop writing when time is called. No exceptions.
(5) Scratch space is provided at the back. You may not use your own scratch paper. If you need more scratch paper, ask Professor Nevins.

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Problem 1. (5 points) Let $x$ and $y$ be variables, and compute the determinant of
\[
\begin{pmatrix}
1 & 0 & 0 & 2 \\
0 & 3 & 1 & 0 \\
0 & 0 & 1 & x \\
y & 0 & 1 & 0
\end{pmatrix},
\]
showing work that justifies your answer.
\[
\det \begin{pmatrix}
1 & 0 & 0 & 2 \\
0 & 3 & 1 & 0 \\
0 & 0 & 1 & x \\
y & 0 & 1 & 0
\end{pmatrix} = 1 \det \begin{pmatrix}
3 & 0 \\
0 & 1 \\
y & 0
\end{pmatrix} - 2 \det \begin{pmatrix}
0 & 3 \\
0 & 1 \\
x & 0
\end{pmatrix}
= 3 \det \begin{pmatrix}
1 & x \\
1 & 0
\end{pmatrix} - 2y \det \begin{pmatrix}
3 & 1 \\
0 & 1
\end{pmatrix} = -3x - 6y.
\]

Problem 2. (4 points) Let $\beta = \{e_1, e_2, e_3\}$ be the standard ordered basis of $\mathbb{R}^3$, and let $\gamma = \{(1,0,1), (1,1,0), (0,1,1)\}$. Compute the change-of-coordinate matrix $Q = [Q_{\mathbb{R}^3}]_\beta$ from $\beta$ coordinates to $\gamma$ coordinates, showing work that justifies your answer.
\[
(1,0,0) = \frac{1}{2}(1,0,1) + \frac{1}{2}(1,1,0) + \frac{-1}{2}(0,1,1)
\]
\[
(0,1,0) = \frac{1}{2}(1,0,1) + \frac{1}{2}(1,1,0) + \frac{1}{2}(0,1,1)
\]
\[
(0,0,1) = \frac{1}{2}(1,0,1) + \frac{1}{2}(1,1,0) + \frac{1}{2}(0,1,1)
\]

Thus:
\[
Q = \begin{pmatrix}
\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\
-\frac{1}{2} & \frac{1}{2} & \frac{1}{2}
\end{pmatrix}.
\]
Problem 3. (6 points total)

(1) What are the possible values of the nullity of $A$ for a $3 \times 3$ matrix $A$?

$$0, 1, 2, 3$$

(2) Let $A$ be the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & c \\ 0 & 1 & 1 \end{pmatrix}$$

The nullity (dimension of the null space) of $A$ may depend on $c$. Describe for each $c \in \mathbb{R}$ the nullity of the corresponding matrix $A$ above.

If $c \neq 3$, continuing row reduction yields

$$A \xrightarrow{-R_1 + R_2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & c-1 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow{R_2 \rightarrow R_2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & c-1 \end{pmatrix} \xrightarrow{-2R_2 + R_3} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & c-3 \end{pmatrix}$$

If $c \neq 3$, continuing row reduction yields $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & c-3 \end{pmatrix}$ nullity 0.

If $c = 3$, then $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is in RREF, has nullity 1.

Thus:

- nullity is 0 if $c \neq 3$
- nullity is 1 if $c = 3$

Problem 4. (3 points) Consider the matrix $A$ shown below:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 4 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 4 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 2 & 1 & 2 \end{pmatrix}$$

Give an example of a vector not in the range of $A$, with justification.

The range of $A$ is the span of its columns; sixth column is a linear combination of first five columns. Write $v_1, \ldots, v_5$ for the first five columns. Then

$\text{span}\{v_1, \ldots, v_5\} = \text{span}\{e_1, e_2, e_3, (0, 0, 0, 1, 1, 2), (0, 0, 0, 1, 1, 1)\}$

Claim: $(0, 0, 0, 0, 1, 0) \notin \text{span}\{v_1, \ldots, v_5\}$. Indeed, if it equaled

$cv_1 + c_5v_5 = (c_1, c_2, c_3, c_4 + c_5, c_4, 2c_4 + c_5) = (0, 0, 0, 0, 1, 0)$,

we get $c_1 = c_2 = c_3 = 0$, $c_4 = 0$, $c_4 + c_5 = 0 \Rightarrow c_5 = -1$, and $0 = 2c_4 + c_5 = 2(0) + (-1) = -1$, a contradiction. So $(0, 0, 0, 0, 1, 0) \notin \text{range of } A$. 

Problem 5. (9 points total) Define a linear transformation \( T : \mathbb{R}[x]_{\leq 3} \rightarrow \mathbb{R}[x]_{\leq 3} \) by
\[
T(f) = x \frac{\partial f}{\partial x} - 2 \frac{\partial f}{\partial x} + f(0).
\]
(1) Compute the matrix \([T]_{\beta}\) of \( T \) in the ordered basis \( \beta = \{1, x, x^2, x^3\} \).
\[
T(1) = 1 \quad T(x) = x - 2 \quad T(x^2) = 2x^2 - 4x \quad T(x^3) = 3x^3 - 6x^2.
\]
Thus
\[
[T]_{\beta} = \begin{pmatrix}
1 & -2 & 0 & 0 \\
0 & 1 & -4 & 0 \\
0 & 0 & 2 & -6 \\
0 & 0 & 0 & 3
\end{pmatrix}
\]

(2) Is \( T \) injective? [ ] yes [ ] no
Is \( T \) surjective? [ ] yes [ ] no

(3) Justify your answers to part (2).

Since domain & target of \( T \) have same dimension,
\[
\text{nullity}(T) = 0 \quad \text{rank}(T) = \dim \mathbb{R}[x]_{\leq 3}, \quad \text{i.e.}
\]
\( T \) is injective if & only if \( T \) is surjective.

So, suppose \( T(a_0 + a_1 x + a_2 x^2 + a_3 x^3) = 0 \). Then, computing
\[
T(a_0 + a_1 x + a_2 x^2 + a_3 x^3) = (a_0 - 2a_1) + (a_1 - 4a_2)x + (2a_2 - 6a_3)x^2 + 3a_3 x^3,
\]
we get
\[
\begin{align*}
a_0 - 2a_1 &= 0 \\
a_1 - 4a_2 &= 0 \\
2a_2 - 6a_3 &= 0 \\
3a_3 &= 0
\end{align*}
\]
Thus, \( N(T) = \{0\} \), proving that \( T \) is injective,

Hence also surjective.
Problem 6. (3 points) Suppose that $S : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ is a function such that $S(1, 1) = (1, 3, 4)$, $S(1, -1) = (1, 1, -2)$, $S(2, 4) = (2, 8, 12 + c)$. For which values of $c$ could $S$ be a linear transformation? Justify.

Since $3(1, 1) + (-1)(1, -1) = (2, 4)$, if $S$ is a linear transformation we get

$$S(2, 4) = S(3(1, 1) + (-1)(1, -1)) = 3S(1, 1) - S(1, -1)$$

or

$$(2, 8, 12 + c) = 3(1, 3, 4) - (1, 1, -2)$$

$$= (3, 9, 12) - (1, 1, -2) = (2, 8, 14).$$

Thus $S$ can only be linear if $12 + c = 14$, i.e., $c = 2$.

Problem 7. (4 points) Consider the linear transformation $T : M_{2 \times 2}(\mathbb{R}) \rightarrow M_{2 \times 2}(\mathbb{R})$ defined by $T(A) = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} A$ (matrix multiplication). Find a basis for the range $\mathcal{R}(T)$. Justify your answer.

$$T \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 2a + c & 2b + d \\ 0 & 0 \end{pmatrix}.$$ 

In particular, $T \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Thus

$$\mathcal{R}(T) = \left\{ \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} \mid c, d \in \mathbb{R} \right\}.$$ 

This subspace is spanned by $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}$, which is also a basis for $\mathcal{R}(T)$—indeed, if $c_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

then $c_1 = 0 = c_2$.

So these vectors are linearly independent.

Concluding, $\mathcal{R}(T)$ has basis $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}$.
Problem 8. (5 points) Let $V$ be a finite-dimensional vector space with identity linear transformation $I : V \to V$ (that is, $I(v) = v$ for all $v \in V$).

Suppose that $T : V \to V$ is a linear transformation with $T^n = 0$ (the zero linear transformation) for some $n \geq 1$. Prove that $I - T$ is invertible. [Hint: consider the null space $\mathcal{N}(I - T)$.

Suppose $V$ is finite-dimensional and

$$T^n = \underbrace{T \circ T \circ \ldots \circ T}_{n \text{ times}} = 0 \quad \text{for some } n \geq 1.$$

Suppose $v \in \mathcal{N}(I-T)$, i.e. $(I-T)(v) = 0$, or equivalently $v - T(v) = 0 \Leftrightarrow T(v) - T(v) = 0$, i.e. $v = T(v)$. Then

$$0 = 0 \cdot v = \underbrace{T(T \ldots (T(v))}_{n \text{ times}} = T(T \ldots T(v)) = \ldots = T(v) = v,$$

i.e. $v = 0$. Thus $\mathcal{N}(I-T) = \{0\}$, so $I-T$ is injective. By the Dimension Theorem, we also have

$$\dim V = \dim \mathcal{N}(I-T) + \dim \mathcal{R}(I-T) = 0 + \dim \mathcal{R}(I-T),$$

so $\mathcal{R}(I-T) = V$, i.e. $I-T$ is surjective.

Thus, $I-T$ is an isomorphism, as desired.

[Alternatively, you can use $T^n = 0$ to compute

$$(I-T)(I+T+T^2+\ldots+T^{n-1})$$

$$= I - T + T - T^2 + T^2 - \ldots + T^n = I - T^n = I$$

$$= (I+T+\ldots+T^{n-1})(I-T),$$

so

$$(I-T)^{-1} = I+T+\ldots+T^{n-1}.$$]