Math 416 Homework 8

Problem 1.
(1) Suppose \( E \) is the elementary matrix obtained from \( I_n \) by the row operation \( R \), that is, \( I_n \xrightarrow{R} E \).

Prove that for all \( A \in M_{n \times n}(\mathbb{R}) \) one has \( A \xrightarrow{R} EA \). Said another way, left-multiplication by \( E \) implements the row operation that built \( E \) in the first place.

(2) Prove the corresponding statement for elementary column operations.

Problem 2. A matrix \( Q \in M_{n \times n}(\mathbb{R}) \) is called orthogonal if \( QQ^t = I_n \).

(1) Prove that if \( Q \) is orthogonal then \( \det(Q) = \pm 1 \).

(2) Give examples of orthogonal matrices for \( n = 2 \) with both possible values of the determinant.

Problem 3. Suppose \( A, B \in M_{n \times n}(\mathbb{R}) \) satisfy \( AB = I_n \).

(1) Use the determinant to prove that \( A \) is invertible.

(2) Prove or disprove: \( B = A^{-1} \).

Problem 4. In Section 5.1 of [FIS], do Problem 2 parts (a) and (c).

If \( T : V \to V \) is a linear transformation (called a linear operator on \( V \)), an eigenvector of \( T \) with eigenvalue \( \lambda \) (a scalar) is a nonzero vector \( v \) for which \( Tv = \lambda v \). We say \( \lambda \) is an eigenvalue of \( T \) if there is a nonzero vector \( v \in V \) that is an eigenvector of \( T \) with eigenvalue \( \lambda \).

Problem 5. Let \( T \) be a linear operator on a finite-dimensional vector space \( V \).

(1) Show that \( T \) is invertible if and only if 0 is not an eigenvalue of \( T \).

(2) If \( T \) is invertible, show that \( \lambda^{-1} \) is an eigenvalue of \( T^{-1} \) if and only if \( \lambda \) is an eigenvalue of \( T \).

Problem 6. Suppose \( T : V \to V \) is a linear operator with \( V \) finite-dimensional. Suppose \( v \in V \) is an eigenvector of \( T \) with eigenvalue \( \lambda \). As usual, \( T^m : V \to V \) denotes composition of \( T \) with itself \( m \) times. Prove that \( v \) is also an eigenvector for \( T^m \) and give a formula for the corresponding eigenvalue.

Problem 7. In Section 5.1 of [FIS], do Problem 3(a).

Problem 8. In Section 5.1 of [FIS], do Problem 4 parts (b) and (h).