Math 416 Homework 3

Problem 1. Is it possible to write the vector \((1, 0, 3) \in \mathbb{R}^3\) as a linear combination of the vectors \((1, 2, 2)\) and \((1, 3, 4)\)? Justify your answer.

Problem 2. (1) Prove that the set \(\mathbb{R}[x]\) of polynomials \(f(x)\) with real coefficients is a vector space (over the field \(\mathbb{R}\)), where addition of vectors is the usual addition of polynomials, and scalar multiplication is the usual multiplication of a polynomial by a real number.

(2) Recall that the degree of a nonzero polynomial \(f(x)\) is the largest \(d\) for which \(x^d\) appears in \(f(x)\) with a nonzero coefficient. So, for example, \(x^2 - 1\) has degree 2, and \(2x^3 - 3x + 1\) has degree 3 (and \(f(x) = 1 = 1x^0\) has degree 0). Prove that, for each \(n \geq 0\), the subset \(W_n = \{f(x) \in \mathbb{R}[x] \mid \deg(f(x)) \leq n\} \cup \{0\}\) is a subspace of \(\mathbb{R}[x]\).

Problem 3. In class, I defined the span of a finite list of vectors \(u_1, u_2, \ldots, u_n\). More generally, given a nonempty subset \(S\) of a vector space \(V\), one defines \(\text{span}(S)\) to be the set of all linear combinations of vectors in \(S\). Here are some problems about the span.

(1) Section 1.4 of [FIS], Problem 5: parts (g) and (h).

(2) Suppose \(S_1\) and \(S_2\) are subsets of a vector space \(V\). Show that if \(S_1\) is contained in \(S_2\), then \(\text{span}(S_1)\) is contained in \(\text{span}(S_2)\).

(3) Let \(V = \mathbb{R}^2\) and \(S = \{(x, y) \mid x \geq 0 \text{ and } y \geq x\}\). Find \(\text{span}(S)\).

Problem 4. Solve each of the following linear systems by writing down its augmented matrix, doing row operations to get a matrix in reduced row echelon form, and using that to find all of the solutions. You should label your row operations as in §RREF of [B].

(1)
\[
\begin{align*}
2x_1 + x_2 &= 0 \\
x_1 + x_2 &= 1 \\
3x_1 + 4x_2 &= 5 \\
3x_1 + 5x_2 &= 7
\end{align*}
\]

(2)
\[
\begin{align*}
y_1 + 2y_2 - y_3 &= 1 \\
y_1 + y_2 + 2y_3 &= 0 \\
5y_1 + 8y_2 + y_3 &= 1
\end{align*}
\]

(3)
\[
\begin{align*}
2x_1 + 4x_2 + 5x_3 + 7x_4 &= 18 \\
x_1 + 2x_2 + x_3 - x_4 &= 3 \\
4x_1 + 8x_2 + 7x_3 + 5x_4 &= 24
\end{align*}
\]

Problem 5. Suppose that \(A, B,\) and \(C\), are \(m \times n\) matrices with real coefficients. Prove the following three facts using the definition of row equivalence.

(1) \(A\) is row equivalent to \(A\).

(2) If \(A\) is row equivalent to \(B\), then \(B\) is row equivalent to \(A\).

(3) If \(A\) is row equivalent to \(B\), and \(B\) is row equivalent to \(C\), then \(A\) is row equivalent to \(C\).

Note: A relationship that satisfies these three properties is known as an equivalence relation; this is a formal way of saying that a relationship behaves like equality, without requiring the relationship to be as strict as equality itself.

Problem 6. Suppose \(A\) is an \(m \times n\) matrix with real entries. The null space of \(A\), denoted \(\text{N}(A)\), is the set of all solutions in \(\mathbb{R}^n\) to the linear system \(\text{LS}(A, 0)\), where here \(0\) is the zero vector in \(\mathbb{R}^m\). Prove that \(\text{N}(A)\) is a subspace of \(\mathbb{R}^n\).