HW 3

# 1 Try to solve \( c_1(1, 2, 2) + c_2(1, 3, 4) = (1, 0, 3) \), i.e.

\[
\begin{align*}
\begin{bmatrix}
    c_1 + c_2 = 1 \\
    2c_1 + 3c_2 = 0 \\
    2c_1 + 4c_2 = 3 \\
\end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
\begin{bmatrix}
    1 & 1 & 1 \\
    2 & 3 & 0 \\
    2 & 4 & 3 \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
    1 & 1 & 1 \\
    0 & 1 & -2 \\
    0 & 2 & 1 \\
\end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
R_2 + R_3 & \rightarrow (0 \ 1 \ -2) \sim x_2 = -2 \\
-R_2 + R_1 & \rightarrow (0 \ 0 \ 5) \sim x_1 = 3, \quad x_3 = 5
\end{align*}
\]

no solution!

So \((1, 0, 3)\) cannot be so written.

# 2 (1) \[ R[x] \subseteq \text{Fun}(\mathbb{R}, \mathbb{R}) \], with addition of polynomials and scalar multiplication agreeing with those in \[ \text{Fun}(\mathbb{R}, \mathbb{R}) \], so \[ R[x] \] is a vector space if and only if it is a subspace of \[ \text{Fun}(\mathbb{R}, \mathbb{R}) \].

But: (1) the zero function is a polynomial,

(2) the sum of polynomials is a polynomial,

(3) a scalar multiple of a polynomial is a polynomial,

so \[ R[x] \] is a subspace, hence a vector space.

(2) Wu contains zero by definition. Also

\[
(c_0 + a_1x + \cdots + a_nx^n) + (b_0 + \cdots + b_nx^n) = (c_0 + b_0) + \cdots + (c_n + b_n)x^n,
\]

so Wu is closed under addition. Similarly

\[
c(c_0 + a_1x + \cdots + a_nx^n) = (cc_0) + \cdots + (cc_n)x^n,
\]

so Wu is closed under scalar multiplication. Hence Wu is a subspace.
\[ \begin{align*}
(1, 2) = c_1 (1, 0) + c_2 (0, 1) + c_3 (1, 1) \\
= (c_1 + c_3, c_2 + c_3) \Rightarrow c_1 = 3, c_2 = 4
\end{align*} \]

\[ \Rightarrow 3 + c_3 = 1, \quad 4 + c_3 = 2 \Rightarrow c_3 = -2. \]

Get \[ 3 \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} + 4 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix} \]

as required.

\[ \begin{align*}
(1, 0) = c_1 (1, 0) + c_2 (0, 1) + c_3 (0, 0) \\
= (c_1 + c_3, c_2 + c_3) \Rightarrow c_1 = 0, c_2 = 0, 1
\end{align*} \]

\[ \Rightarrow c_3 = 1, \quad 1 + c_3 = 0, \text{ impossible!} \]

So not in the span.

(2) If \( S_1 \subseteq S_2 \) and \( v_1, \ldots, v_k \in S_1 \), \( c_1, \ldots, c_k \) scalars, then \( v_1, \ldots, v_k \in S_2 \) implies
\[ c_1 v_1 + \cdots + c_k v_k \in \text{span}(S_2). \]

So every linear combination of elements of \( S_1 \) lies in \( \text{span}(S_2) \), i.e. \( \text{span}(S_1) \subseteq \text{span}(S_2) \).

(3) \( (0, 1), (1, 1) \in S \). If \( (a, b) \in \mathbb{R}^2 \), then
\[ (a, b) = a (1, 1) + (b-a) (0, 1). \]

So \( \mathbb{R}^2 \subseteq \text{span}\{ (0, 1), (1, 1) \} \subseteq \text{span}(S) \).

Thus \( \text{span}(S) = \mathbb{R}^2 \).
(1) Performing no row operations on $A$ yields $A$, so $A$ is row equivalent to $A$.

(2) It suffices to show that if applying a single row operation to $A$ gives $B$, then a sequence of row operations applied to $B$ gives $A$.

- Switching rows $R_i$ and $R_j$: applying the same operation to $B$ returns $A$.
- Rescaling row $R_i$ by multiplication by $c 
eq 0$: rescaling row $R_i$ of $B$ by multiplication by $\frac{1}{c}$ returns $A$.
- Adding row $R_i$ to row $R_j$: the sequence of operations:
  - Rescale $R_i$ by mult. by $(-1)$
  - Add row $i$ to row $j$
  - Rescale $R_i$ by mult. by $(-1)$

applied to $B$ returns $A$.

Thus if $A$ is row equivalent to $B$, then $B$ is row equivalent to $A$.

(3) If a sequence $S_1, \ldots, S_k$ of row operations applied to $A$ gives $B$, and a sequence $T_1, \ldots, T_k$ of row operations applied to $B$ gives $C$, then sequence $S_1, \ldots, S_k, T_1, \ldots, T_k$ applied to $A$ gives $C$.

Thus $A$ row equivalent to $B$ and $B$ row equivalent to $C$ gives $A$ row equivalent to $C$. 

Suppose $LS(A,0)$ consists of equations
\[ a_{i1}x_1 + \ldots + a_{in}x_n = 0, \quad 1 \leq i \leq m. \]

(i) $(x_1, \ldots, x_n)$ is a solution of $LS(A,0)$ so $(0, \ldots, 0) \in \text{Null}(A)$.

(ii) If $(x_1, \ldots, x_n) \in \text{Null}(A)$ and $(y_1, \ldots, y_n) \in \text{Null}(A)$, then
\[
\begin{align*}
q_{i1}(x_1 + y_1) + \ldots + q_{in}(x_n + y_n) &= q_{i1}x_1 + \ldots + q_{in}x_n + \\
&\quad + q_{i1}y_1 + \ldots + q_{in}y_n = 0 + 0 = 0.
\end{align*}
\]

So $(x_1, \ldots, x_n) + (y_1, \ldots, y_n) \in \text{Null}(A)$.

(iii) If $(x_1, \ldots, x_n) \in \text{Null}(A)$, and $c$ is a scalar, then
\[
\begin{align*}
0 &= c \cdot (q_{i1}x_1 + \ldots + q_{in}x_n) \\
&= q_{i1}(cx_1) + \ldots + q_{in}(cx_n) \\
&= c(x_1, \ldots, x_n) = (cx_1, \ldots, cx_n) \in \text{Null}(A),
\end{align*}
\]

Thus $\text{Null}(A)$ satisfies the three required properties to be a subspace.