The Cubic Formula

Problem 1. Suppose $f(x) = x^3 + ax^2 + bx + c$ is a cubic polynomial (coefficients in a field K with $\mathbb{Q} \subseteq K \subseteq \mathbb{C}$). Show that substituting $x = X - a/3$ gives a polynomial $f(X)$ with zero quadratic term. □

Next consider a cubic polynomial $f(x) = x^3 + px + q$ (we assume we have already changed variables as above). Suppose the roots of $f(x)$ are complex numbers $\alpha_1, \alpha_2, \alpha_3$.

Problem 2. Show that

$\alpha_1 + \alpha_2 + \alpha_3 = 0, \qquad \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3 = p, \qquad \alpha_1\alpha_2\alpha_3 = -q.$ □

Now consider two new variables u and v.

Problem 3. Show that if

$$(v^3 - u^3) + q = 0 \text{ and } 3uv = p,$$

then $f(v - u) = 0$. Then, use the second equation to eliminate u from the first equation and show:

$$v^3 - \frac{p^3}{27v^3} + q = 0.$$

□

From the previous formula, we clear denominators and consider it as a *quadratic* formula for v^3; it has solutions

$$v^3 = -\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}.$$

Let $\omega = e^{2\pi i/3}$ (a complex cube root of 1), and let A denote a cube root of

$$-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}.$$

Problem 4. Explain why the solutions for v are A, ωA, and $\omega^2 A$, and thus the roots of $f(x) = x^3 + px + q$ are

$$\alpha_1 = A - \frac{p}{3A}, \qquad \alpha_2 = \omega A - \omega^2 \frac{p}{3A}, \qquad \alpha_3 = \omega^2 A - \omega \frac{p}{3A}.$$

□

Whew! That was complicated. Here's something more pleasant. Recall that the *discriminant* of $f(x) = x^3 + px + q$ is

$$\delta = (\alpha_2 - \alpha_1)(\alpha_3 - \alpha_1)(\alpha_3 - \alpha_2) = \det \begin{pmatrix} 1 & 1 & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_1^2 & \alpha_2^2 & \alpha_3^2 \end{pmatrix}.$$

Problem 5.

(1) Check that $\delta^2 = -4p^3 - 27q^2$.

Now consider the function $F(x, y) = y^2 - x^3 - px - q$.

(2) Suppose that $\delta^2 \neq 0$. Show that there are no common solutions (a, b) to $F(a, b) = 0$, $\partial F/\partial x(a, b) = 0$, and $\partial F/\partial y(a, b) = 0$.