Rational Functions

Let \( V \subseteq \mathbb{A}^n_k \) be a variety. Then \( \Gamma(V) \) is a domain and we may form

\[ k(V) \triangleq \text{Frac}(\Gamma(V)) \]

the field of rational functions on \( V \).

If \( f \in k(V) \) and \( a \in V \), then \( f \) is defined at \( a \) if \( \exists p,q \in \Gamma(V) \) with \( f = \frac{p}{q} \) and \( q(a) \neq 0 \).

**Example** Let \( V = V(xy - zw) \subseteq \mathbb{A}^4_k \).

Then \( f = \frac{x}{w} - \frac{z}{y} \) in \( k(V) \), so \( f \) is defined on \( V \setminus (V \cap V(\bar{y}) \cap V(\bar{w})) \). [Draw a picture!]

So if \( \Gamma(V) \) is not a UFD, it may not suffice.

\(<\) Here \( \bar{x} \) denotes the image of \( x \in k[x,y,z,w] \)

in \( \Gamma(V) \), etc.
to check for a given reduced fraction $f/g$ whether $g(a) \neq 0$ to check whether $f$ is defined at $a$.

**Def** If $e \in V$,

$$\mathcal{O}_a(V) = \{ f \in \Gamma(V) \mid f \text{ is defined at } a \}.$$  

Equivalently, $\mathcal{O}_a(V)$ can be obtained by "inverting only those functions that do not vanish at $e$." This is the local ring of $V$ at $a$.

A point $a \in V$ at which $f \in k(V)$ is not defined is a **pole** of $f$.

**Prop** (i) The set of poles of $f \in k(V)$ is an algebraic subset of $V$.

(2) $\Gamma(V) = \bigcap_{a \in V} \mathcal{O}_a(V)$.

**Pf.** (1) \[ \{ a \in V \mid f \text{ is not defined at } a \} = \bigcap_{f \in k(V)} V(f). \]

(2) Let $I_f = \{ g \in k[x_1, \ldots, x_n] \mid gf \in \Gamma(V) \}$ (for $f \in k(V)$). Then $I_f \subseteq k[x_1, \ldots, x_n]$ is an ideal, the "ideal of possible denominators of $f$." Note $V(I_f) = \{ a \in V \mid f \text{ is not defined at } a \}.$
If $f \in \bigcap_{a \in V} Q_a(V)$ then $V(f) = \emptyset$ so by Nullstellensatz, $\exists \ f = (1)$. \hfill \Box.

If $f \in Q_a(V)$ then $f(a)$ is well-defined.

**Definition:**

$$M_a(V) = \{ f \in Q_a(V) \mid f(a) = 0 \}.$$  

$$= \ker (\text{ev}_a : Q_a(V) \to k).$$  

$$= \{ f \in Q_a(V) \mid f \text{ is not a unit} \}.$$  

in $Q_a(V)$

**Lemma:** Let $R$ be a ring. Then $R$ has a unique maximal ideal iff the set of non-units of $R$ is an ideal in $R$.

**Proof.** ($\Rightarrow$) Suppose $m \subseteq R$ is the unique maximal ideal. Then all the non-units of $R$ lie in $m$, and if $x \in R \setminus m$ then $(x) = R \Rightarrow x$ is a unit.

($\Leftarrow$) Let $m$ be the set of non-units. This contains every proper ideal. \hfill \Box.

Cor. (Q_a(V)) is a local ring.
**Definition** A ring $R$ that has a unique maximal ideal is called a **local ring**.

**Origin of the term** The ring $O_a(V)$ only "sees what is happening near $a".

**Example** There is a one-to-one correspondence between prime ideals of $O_a(V)$ and subvarieties of $V$ that contain $a$.

**Remark** Compare to the construction of the homwork: $k[x_1,...,x_n,g^{-1}]$ is the coordinate ring of $\mathbb{A}^n_k \setminus V(g)$. Inverting $g$ means cutting out $V(g)$. So, in $k(V)$, inverting everything means "cutting out every point." And in $O_a(V)$, inverting everything except those that vanish at $a$ means "cutting out everything but $a"."}
Example  Our homomorphism

\[ k[x,y]/(y^2-x^3) \rightarrow k[t] \]

before induces an isomorphism on fields of rational functions.

Proof. For any \( a \in V \), any \( V \), \( \mathcal{O}_a(V) \) is a Noetherian ring.

Pf. Given \( I \subseteq \mathcal{O}_a(V) \), let

\[ J = I \cap \Gamma(V) \subseteq \mathcal{O}_a(V) \]. Then \( J \) is an ideal of \( \Gamma(V) \). Furthermore, if \( f = \frac{p}{q} \in I \) then \( q \cdot f = p \in J \); so

\[ \mathcal{O}_a(V) \cdot J = \left\{ g \cdot h \mid g \in \mathcal{O}_a(V), h \in J \right\} \]

= \( I \).

Writing \( J = (f_1, \ldots, f_r) \) gives

\[ I = \mathcal{O}_a(V) \cdot (f_1, \ldots, f_r) = (f_1, \ldots, f_r) \subseteq \mathcal{O}_a(V) \]. \( \square \)

* In this example, we needn't have taken "ideal generated by" since this is already an ideal. But in general, if you have a homomorphism \( R \rightarrow S \) and an ideal \( I \subseteq R \), then
already an ideal. But in general, if you have a homomorphism \( R ightarrow S \) and an ideal \( J \subseteq R \), then

\[
S \cdot J = \left\{ s \cdot j \mid s \in S, j \in J \right\}.
\]
A Detour into Projective Space

It's convenient, as we'll discuss in more detail later, to "enlarge" affine space a little.

Def. The projective space $\mathbb{P}_k^n$ is

$$\mathbb{P}_k^n = \{\text{lines } L \subseteq k^{n+1}\}$$

$$= \frac{k^{n+1} \setminus 0^n}{k^*} \overset{def}{=} \{v \in k^{n+1} \mid 0^n \setminus v\} / k^*$$

where $v \sim w$ if $\exists \lambda \in k$, $\lambda \neq 0$, s.t. $v = \lambda w$. This is an equivalence relation on $k^{n+1} \setminus 0^n$.

Lemma. Taking $v \in k^{n+1} \setminus 0^n$ to

$$L = \{\lambda v \mid \lambda \in k\} \subseteq k^{n+1}$$

gives a bijection from the set of equivalence classes to the set of lines in $k^{n+1}$ (note "line" means "linear subspace of dimension 1").

Notation. The equivalence class of
Notation The equivalence class of $v \in \mathbb{F}^{n+1}$ is written $[v]$ or, if $v = (v_0, \ldots, v_n)$, as $(v_0 : \ldots : v_n)$ ("homogeneous coordinates" on $\mathbb{P}^n$).
Example \( \mathbb{P}^1_{\mathbb{R}} = \{ \text{lines in } \mathbb{R}^2 \} \uparrow \), \( S^1/\sim \)

where \( p \sim q \) if \( p \) and \( q \)
are antipodal points, i.e., \( p = -q \).

\[ S^1 \equiv S^1. \]

Example \( \mathbb{P}^1_{\mathbb{C}} \). This is harder to draw.

First, there's a subset
\[ U \subseteq \mathbb{P}^1_{\mathbb{C}} = \{ L \subseteq \mathbb{C}^2 \} \]
\[ = \{ L = \mathbb{C} \cdot v \mid v = (a, b) \text{ with } b \neq 0 \}. \]
If \( L = \mathbb{C} \cdot v \) w/ \( b \neq 0 \), then 
\[
L = \mathbb{C} - \frac{1}{b} \cdot v = \mathbb{C} \cdot \left( \frac{a}{b}, 1 \right).
\]
Each such line has (furthermore) a unique vector \( w \in L \) with \( w = (c, 1) \).

So there is a bijection 
\[
1 \rightarrow u
\]
\[
c \rightarrow c(c, 1)
\]

What did we leave out? (Only)
\[
l = \{ (c,0) \mid c \in \mathbb{C} \}.
\]
So, 
\[
P^1 \setminus \mathbb{C} = \mathbb{C} \cup \{ \mathbb{C} \} = \mathbb{P}^1 \cup \{ \infty \}.
\]
where \( \infty \) denotes the line \( l \) above.

\[
\mathbb{C} \quad \rightarrow \quad \mathbb{P}^1
\]
Note that $P^1 \mathcal{C}$ actually has two obvious “copies of $\mathcal{A}_c = \mathcal{C}$” sitting inside it: the other is

$$U' = \left\{ \left. L = \mathcal{C} \cap \mathcal{V} \right| v = (a/b), a \neq 0 \right\}.$$

$$= \left\{ L = \mathcal{C} \cdot (1, c) \mid c \in \mathbb{C} \right\}.$$

This omits $L_0 = \mathcal{C} \cdot (0, 1)$ and has $L_0$ at its origin!
So, we obtain \( P^n \) (or indeed any \( P^k \)) by gluing together two copies of \( A^1 = \mathbb{C} \).

In general, \( P^n_k \) is obtained by gluing together \( n+1 \) copies — with lots of overlaps! — of \( A^n_k \).

Always have

\[
\mathcal{U}_n = \mathcal{P}_n \backslash U_n = \left\{ L = k \cdot v \leq k^n \mid v = (v_0, \ldots, v_n), \ v_n = 1 \right\}
\]

\[
\mathcal{U}_n \ni \mathcal{P}_n \backslash \mathcal{U}_n = \left\{ L = k \cdot v \leq k^n \mid 0 \neq v = (v_0, \ldots, v_{n-1}, 0) \right\}
\]

\[
\mathcal{P}_n \backslash \mathcal{U}_n = \left\{ L = k \cdot v \leq k^n \mid v \neq 0 \right\} = \mathcal{P}_{n-1}^n\!. \]
let's come back to this in a few minutes. First, note:

**Fact** Suppose \( f \in k[x_0, \ldots, x_n] \) is a homogeneous polynomial of degree \( d \), i.e., \( f(\lambda \cdot x) = \lambda^d f(x) \) \( \forall \lambda \in k \).

[i.e. all monomials appearing in \( f \) have degree exactly \( d \).]

\( f \) does not determine a function \( \mathbb{P}^n_k \to k \), since

\[ f(\lambda \cdot v) \neq f(v) \text{ in general (but } \quad [v] = [\lambda v] \text{ if } \lambda \neq 0 \text{).} \]

But,

\[ V(f) = \{ v \in \mathbb{P}^n_k \mid f(v) = 0 \} \text{ does!} \]

After all, for \( \lambda \neq 0 \),

\[ f(v) = 0 \text{ iff } f(\lambda v) = \lambda^d f(v) = 0. \]
Now, let's return to an example, the projective plane $\mathbb{P}^2_k$. Consider a line $l \subseteq \mathbb{P}^2_k$ given as the zero set of a homogeneous linear equation,

$$l = \mathcal{V}(ax + by + cz).$$

Then

$$l \cap U_2 = \left\{(x:y:z) \mid z \neq 0, \ ax + by + cz = 0 \right\}$$

$$\cong \left\{(x:y) \in \mathbb{A}^2_k \mid ax + by + c = 0 \right\}.$$}

and

$$l \cap (\mathbb{P}^2_k \setminus U_2) = \left\{(x:y:z) \mid z = 0, \ ax + by + cz = 0 \right\}$$

$$= \left\{(x:y:0) \mid ax + by = 0 \right\}.$$}

$$= \left\{(-b:a:0) \right\} \in \mathbb{P}^1_k.$$
let's suppose $b \neq 0$. Then

$ln U^2$ is a line in the plane with

slope $-\frac{a}{b}$, and $ln \mathbb{P}^1_K$ is

the point $(1 : \text{slope})$.

**Conclusion.** $\mathbb{P}^1_K$ is exactly $\mathbb{A}^2_K$ union

the set of slopes of lines in $\mathbb{A}^2_K$.

[Remark. There is one more point of $\mathbb{P}^1_K$

which corresponds to lines of infinite

slope].

**Terminology.** $\mathbb{P}^2_K \setminus U^2 \cong \mathbb{P}^1_K$ is called

the "line at infinity in $\mathbb{P}^2_K$."