Math 428, Homework 5

Problem 1. Fix a field $k$. Let $f_1, \ldots, f_r, g \in k[x_1, \ldots, x_n]$ be polynomials with $g$ nonzero. Let $k(x_1, \ldots, x_n)$ denote the field of rational functions in variables $x_1, \ldots, x_n$ (with coefficients in $k$). Let $R = k[x_1, \ldots, x_n, g^{-1}]$ denote the subset of $k(x_1, \ldots, x_n)$ consisting of rational functions that can be represented as fractions of the form $\frac{f}{g^N}$ for some $N \geq 0$ and some $f \in k[x_1, \ldots, x_n]$.

1. Prove that $k[x_1, \ldots, x_n, g^{-1}]$ is a subring of $k(x_1, \ldots, x_n)$.
2. Construct an isomorphism from $Q = k[x_1, \ldots, x_n, x_{n+1}]/(x_{n+1}g-1)$ to $R = k[x_1, \ldots, x_n, g^{-1}]$.
   [Hint: take $x_{n+1}$ to $1/g$.]
3. Prove that the following are equivalent:
   a) There exist $a_1, \ldots, a_r \in k[x_1, \ldots, x_n]$ such that the image of $a_1 f_1 + \cdots + a_r f_r$ in $Q$ is 1.
   b) There exist $c_1, \ldots, c_r \in R$ such that $c_1 f_1 + \cdots + c_r f_r = 1$ in $R$.

Problem 2. Use the results of the previous problem to give a (somewhat) different proof of the Nullstellensatz (assuming, of course, that $k$ is algebraically closed) as follows:

1. Suppose that $g \in V(f_1, \ldots, f_r)$. Use the Weak Nullstellensatz to prove that the ideal in $Q$ generated by the images of $f_1, \ldots, f_r$ is all of $Q$. [This is almost identical to what we did before.]
2. Use part (3) of the previous problem to conclude that there exist rational functions $c_1, \ldots, c_r$ in $R$ such that $c_1 f_1 + \cdots + c_r f_r = 1$ in $R$.
3. Now, clear $g$ from the denominators of $c_1, \ldots, c_r$ and conclude that there are a positive integer $N$ and polynomials $u_1, \ldots, u_r$ such that $u_1 f_1 + \cdots + u_r f_r = g^N$ in $k[x_1, \ldots, x_n]$.

Problem 3. Let $g \in k[x_1, \ldots, x_n]$ be a nonzero polynomial. Let $V = V(x_{n+1}g-1) \subset A_k^{n+1}$. Define a function $\phi : V \to A_k^n$ by $\phi(a_1, \ldots, a_{n+1}) = (a_1, \ldots, a_n)$. Prove that $\phi$ is injective and that its image is exactly $A_k^n \setminus V(g)$.

Problem 4. Do the problem explained in the proof on the last page of the Week 4 notes (on the web).

Problem 5. Let $V = A_k^1$ (with coordinate $t$, so $\Gamma(V) = \mathbb{C}[t]$) and $W = V(y^2 - x^3) \subset A_k^2$.

a) Prove that the polynomial map $\Phi : V \to A_k^2$ given by $\phi(t) = (t^2, t^3)$ has $W$ as its image, and that in fact this gives a bijection from $V$ to $W$.
Let $\phi : V \to W$ be the induced polynomial map from $V$ to $W$.

b) Prove that the induced map $\phi^* : \Gamma(W) \to \Gamma(V)$ is injective.

c) Identify the image of $\phi^*$ (note that it is a subring of $\mathbb{C}[t]$).