(1) Describe the prime ideals in $\mathbb{R}[x]$. Describe the prime ideals in $\mathbb{C}[x]$. Explain why it is difficult to list the prime ideals in $\mathbb{Q}[x]$.

(2) Describe the set of units in the ring $\mathbb{Z}[\sqrt{-p}]$ where $p > 1$ is a prime integer.

(3) Prove that the units in the ring $\mathbb{Z}[\sqrt{7}]$ are the elements of the form $a + b\sqrt{7}$ where $a^2 - 7b^2 = \pm 1$. Show that $8 + 3\sqrt{7}$ is a unit in $\mathbb{Z}[\sqrt{7}]$.

(4) Let $\mathbb{Z}[\sqrt{7}]^\times$ denote the set of units in $\mathbb{Z}[\sqrt{7}]$. Define a function $\phi : \mathbb{Z}[\sqrt{7}]^\times \to \mathbb{R}^2$ by $\phi(a + b\sqrt{7}) = (\log|a + b\sqrt{7}|, \log|a - b\sqrt{7}|)$. Prove that it is a group homomorphism (where $\mathbb{R}^2$ has its structure as an additive group). Then prove that

$$\text{Im}(\phi) \subset \{(x, y) \in \mathbb{R}^2 : x + y = 0\} \cong \mathbb{R}.$$ 

Next, prove that the set of nonzero elements in $\text{Im}(\phi) \subset \mathbb{R}$ contains a smallest element. Use this to prove that $\text{Im}(\phi)$ is an infinite cyclic group. Show that $\Phi(8 + 3\sqrt{7})$ is a generator of this group. Conclude that

$$\mathbb{Z}[\sqrt{7}]^\times = \{\pm(8 + 3\sqrt{7})^k : k \in \mathbb{Z}\}.$$ 

(5) Let $f \in F[x]$ be an irreducible polynomial of degree $n$ ($F$ is a field), and let $E/F$ be a splitting field of $f$ over $F$. Prove that $n$ divides $[E : F]$. Prove that if $f$ is separable, then $n$ divides $|\text{Gal}(E/F)|$.

(6) Let $f \in F[x]$, let $E/F$ be a splitting field. Prove that if $f$ is irreducible over $F$, then $\text{Gal}(E/F)$ acts transitively on the set of roots of $f$.

(7) Do exercises 82, 83, 84, and 85 of Rotman’s *Galois Theory* (page 75). Using your solutions, write out a complete proof of Lemma 73 of Rotman (page 72).