(1) Prove that if $H$ is a subgroup of index 2 of a group $G$, then $H$ is normal in $G$.

(2) Prove that if $n \geq 5$, the alternating group $A_n$ has no subgroups of index $k$ where $2 \leq k \leq n-1$.

(3) Prove that if $G$ is a nonabelian group of order 6, then $G$ is isomorphic to $D_3$, the dihedral group of degree 3, as follows [this fact is proven in many books, so please do not look at the proofs there! just do it yourself]:
   (a) Prove that $G$ contains an element $b$ of order 2.
   (b) Prove that $G$ contains an element $a$ of order 3 by showing that if all elements of a group have order at most 2 then the group is abelian.
   (c) Prove that $bab^{-1} = a^{-1}$ [Hint: why is $\langle a \rangle$ normal in $G$?]
   (d) Use the above to prove that there is an isomorphism $D_3 \to G$.

(4) Prove that $GL_2(\mathbb{F}_2)$ is isomorphic to $S_3$.

(5) Prove that, for $n \geq 3$, every $\sigma \in A_n$ is a product of 3-cycles. [Hint: Show that $(1\ 2\ 3)$ and $(i\ j\ k)$ are conjugate by first treating the case when they are not disjoint (so the two permutations together move at most 5 letters) and then the case when they are disjoint.]

(6) Prove that if a normal subgroup $H$ of $A_n$ contains a 3-cycle, then $H = A_n$. [Hint: For $n = 5$ this is Lemma 2.109 of Rotman.]

(7) Let Aut($G, X$) denote the set of all actions of $G$ on $X$, that is, all functions $G \times X \to X$ satisfying the axioms for a group action. Construct a bijection

\[ \text{Hom}_{\text{Groups}}(G, \text{Symm}(X)) \to \text{Act}(G, X) \]

such that, if $H \to G$ is a homomorphism of groups and $X$ is a $G$-set, there is a natural commutative diagram

\[ \begin{array}{ccc}
\text{Hom}_{\text{Groups}}(G, \text{Symm}(X)) & \to & \text{Act}(G, X) \\
\downarrow & & \downarrow \\
\text{Hom}_{\text{Groups}}(H, \text{Symm}(X)) & \to & \text{Act}(H, X).
\end{array} \]

(8) Prove that if $H$ is a normal subgroup of a group $G$ such that $H$ and $G/H$ are finitely generated, then $G$ is finitely generated. If $H$ and $G/H$ are finitely presented, must $G/H$ be finitely presented?

(9) Suppose $\sigma$ is an element of $S_n$. Find and prove a formula for the size of the conjugacy class of $\sigma$ in $S_n$. [Hint: first write $\sigma$ as a product of disjoint cycles.]