- Do not open this exam packet until I say *start*.
- Turn off all electronic devices and and put away all items except for a pen/pencil and an eraser.
- Remove hats and sunglasses.
- If you have a question, raise your hand and I will come to you. When you stand up, you are done with your exam.
- Quit working and close this packet when I say *stop*.
- Good luck!

![Cartoon of Newton and Leibniz inventing calculus and discussing derivatives]

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*It is possible to score a total of 105 points on this exam, but your score will be out of 100.
1. (a) (5 points) State the definition of the derivative of \( f(x) \) at the point \( x = a \) (we have learned two definitions; you may state either one).

\[
\frac{df}{dx} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \quad \text{or} \quad \frac{df}{dx} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}
\]

(b) (10 points) Let \( f(x) = 4x - 3x^2 \).

Use the definition of the derivative as a limit to prove that \( f'(x) = 4 - 6x \).

Show each step in your calculation and be sure to use proper terminology at each step.

\[
\frac{df}{dx} = \lim_{h \to 0} \frac{[4(x+h) - 3(x+h)^2] - [4x - 3x^2]}{h}
\]

\[
= \lim_{h \to 0} \frac{4x + 4h - 3x^2 - 6xh - 3h^2 - 4x + 3x^2}{h}
\]

\[
= \lim_{h \to 0} \frac{4h - 6xh - 3h^2}{h}
\]

\[
= \lim_{h \to 0} (4 - 6x - 3h)
\]

\[
= 4 - 6x
\]
2. Recall that a function is continuous at \( x = a \) if the following three conditions are satisfied:

I) \( f(a) \) is defined,

II) the limit as \( x \) approaches \( a \) of \( f(x) \) exists, and

III) the limit as \( x \) approaches \( a \) of \( f(x) \) equals \( f(a) \).

(a) (5 points) Sketch the graph of a function that satisfies I and II, but not III for \( a = 2 \).

\[
\begin{align*}
\lim_{{x \to 2}} f(x) &= 1 & \text{but } f(2) &= 2 \\
\text{so } \lim_{{x \to 2}} f(x) &\neq f(2).
\end{align*}
\]

(b) (10 points) Use the Intermediate Value Theorem to show that the equation

\[ x^4 + x - 3 = 0 \]

has a solution in \((1, 2)\).

(2 pts) \( f(x) = x^4 + x - 3 \) is continuous on \((-\infty, \infty)\)

because it is a polynomial.

(3 pts) \( f(1) = 1^4 + 1 - 3 = -1 < 0 \)

(3 pts) \( f(2) = 2^4 + 2 - 3 = 15 > 0 \)

(2 pts) IVT \( \Rightarrow \) There is a number \( c \) in \((1, 2)\)

such that \( f(c) = 0 \).
3. (15 points) Find all **vertical and horizontal** asymptotes of the function

\[ f(x) = \frac{2x^2 + x - 1}{x^2 - x - 2}. \]

Make sure to show sufficient work and write equations of lines (e.g. \( x = 10 \)), not just numbers.

**Vertical**

\[ x^2 - x - 2 = 0 \]

\[ (x + 1)(x - 2) = 0 \]

\[ x = -1, \ 2 \]

**Check**

\[ \lim_{x \to -1} \frac{2x^2 + x - 1}{x^2 - x - 2} = \lim_{x \to 1} \frac{(2x - 1)(x + 1)}{(x+1)(x-2)} \]

\[ = \lim_{x \to -1} \frac{2x-1}{x-2} = \frac{-3}{-3} = 1 \quad \text{NOT AN ASYMPTOTE} \]

**Check**

\[ \lim_{x \to 2^+} \frac{2x^2 + x - 1}{x^2 - x - 2} = \lim_{x \to 2^+} \frac{2x-1}{x-2} \quad (\to \infty) \]

**Vertical, Asymptote at** \( x = 2 \)

**Horizontal**

\[ \lim_{x \to \infty} \frac{2x^2 + x - 1}{x^2 - x - 2} = Z \]

Since

\[ \frac{2x^2 + x - 1}{x^2 - x - 2} \sim \frac{2x^2}{x^2} \sim Z \quad \text{as} \quad x \to \infty \]

**Horizontal, Asymptote at** \( y = 2 \) (Same as above)
4. (6 points each) Evaluate the following limits without the use of derivatives. For infinite limits, you must state whether the limit is approaching $\infty$ or $-\infty$. An answer of "does not exist" is not sufficient.

(a) \[ \lim_{{x \to 1^+}} \frac{x^2 - 9}{x^2 + 2x - 3} = \lim_{{x \to 1^+}} \frac{(x - 3)(x + 3)}{(x - 1)(x + 3)} = \lim_{{x \to 1^+}} \frac{1}{x - 1} = -\infty \]

(b) \[ \lim_{{x \to 1^+}} \frac{4 - x}{|4 - x|} = \lim_{{x \to 4^+}} \frac{4 - x}{-(4 - x)} = \lim_{{x \to 4^+}} -1 = -1 \]

(c) \[ \lim_{{x \to -1}} \frac{\sqrt{3} + x - \sqrt{3}}{x} = \sqrt{3} - \sqrt{3} = -1 \]
(d) \[ \lim_{x \to \pi/2^+} \frac{\sin(2x)}{\sin(x)} = 0 \]

(e) \[ \lim_{x \to \infty} \arctan \left( \frac{x^3 - x}{x^2 - 6x + 5} \right) \] 

\[ \lim_{x \to \infty} \frac{x^3 - x}{x^2 - 6x + 5} = +\infty \text{ since } \frac{x^3 - x}{x^2 - 6x + 5} \sim \frac{x^3}{x^2} = x \text{ as } x \to \infty \]

\[ \lim_{x \to \infty} \arctan \left( \frac{x^3 - x}{x^2 - 6x + 5} \right) = \lim_{x \to \infty} \arctan (x) = \frac{\pi}{2} \]

(f) \[ \lim_{x \to 0^+} \log_a x \quad (\text{assuming } a > 1) = -\infty \]
5. (3 points each) For each of the following statements, circle True if the statement is always true; otherwise circle False.

(a) \[ \lim_{x \to 1} \frac{x^2 + 8x - 9}{x^2 + 7x - 6} = \frac{\lim_{x \to 1} x^2 + 8x - 9}{\lim_{x \to 1} x^2 + 7x - 6} \]

\[ \lim_{x \to 1} x^2 + 8x - 9 = 0 \]

\[ \lim_{x \to 1} x^2 + 7x - 6 = 2 \]

True\hspace{1cm}False

(b) If \(4x - 9 \leq f(x) \leq x^2 - 4x + 7\) for all \(x \geq 0\) then \(\lim_{x \to 4} f(x) = 7\).

\[ \lim_{x \to 4} 4x - 9 = 7 \]

\[ \lim_{x \to 4} x^2 - 4x + 7 = 7 \]

True\hspace{1cm}False

**SQUEEZE THEOREM**!

(c) If \(f\) has domain \([0, \infty)\) and \(f\) has no horizontal asymptote, then

\[ \lim_{x \to \infty} f(x) = \infty \text{ or } \lim_{x \to \infty} f(x) = -\infty. \]

Counter-Example:

\[ \lim_{x \to \infty} \sin x \text{ DNE} \]

\[ y = 3 \sin x, \quad x \geq 0 \]

True\hspace{1cm}False

6. (6 points) Suppose that \(f'(3) = 7\). Circle the three statements below that must be true.

(a) \(f\) is an even function
(b) \(f\) is an odd function
(c) \(f\) is a one-to-one function
(d) \(f\) is differentiable at 7
(e) \(f\) is differentiable at 3
(f) \(f\) is differentiable at 0
(g) \(f\) is continuous at 7
(h) \(f\) is continuous at 3
(i) \(f\) is continuous at 0
(j) \(\lim_{x \to 7} (f(x) - f(7)) = 0\)
(k) \(\lim_{x \to 3} (f(x) - f(3)) = 0\)
(l) \(\lim_{x \to 0} (f(x) - f(0)) = 0\)

MEANS (BY DEFINITION) THAT \(f\) HAS
A DERIVATIVE AT \(x = 3\)

THM: DIFF. AT 3 \(\Rightarrow\) CONT. AT 3

EQUIVALENT TO THE DEFINITION OF
CONTINUITY: \(\lim_{x \to c} f(x) = f(c)\)
7. (9 points) Suppose that \( \lim_{x \to a} [f(x) + g(x)] = 2 \) and \( \lim_{x \to a} [f(x) - g(x)] = 1 \). Evaluate \( \lim_{x \to a} [f(x)g(x)] \).

\[
\lim_{x \to a} [f(x)g(x)]
\]

\[= 4 \text{ pts} +\]
\[5 \text{ E.C. pts}\]

\[\begin{align*}
\text{Note:} \quad \lim_{x \to a} \left[ f(x) + g(x) \right] &= \lim_{x \to a} f(x) + \lim_{x \to a} g(x) \\
\text{Only when both limits are defined.}
\end{align*}\]

We do not know this yet, so we cannot break up these limits.

\[\text{So,}\]
\[\lim_{x \to a} \left[ f(x) + g(x) \right] + \lim_{x \to a} \left[ f(x) - g(x) \right] = 2 + 1
\]

\[\Rightarrow \lim_{x \to a} f(x) = \frac{3}{2}
\]

Similarly,

\[2 - 1 = \lim_{x \to a} \left[ f(x) + g(x) \right] - \lim_{x \to a} \left[ f(x) - g(x) \right]
\]

\[\Rightarrow \lim_{x \to a} g(x) = \frac{1}{2}
\]

Now we know \( \lim_{x \to a} f(x) \) and \( \lim_{x \to a} g(x) \) exist.

\[\begin{align*}
\lim_{x \to a} [f(x)g(x)] &= \lim_{x \to a} f(x) \lim_{x \to a} g(x) \\
&= \frac{3}{2} \cdot \frac{1}{2} = \frac{3}{4}
\end{align*}\]