No calculators allowed.

Show sufficient work to justify each answer.

You have 20 minutes for this quiz.

1. (3 points) Find all critical points of the function \( f(x, y) = yxe^{-(x^2+y^2)} \).

\[
\frac{\partial f}{\partial x} = ye^{-(x^2+y^2)} - 2x^2 ye^{-(x^2+y^2)} = 0 \quad \Rightarrow \quad y - 2x^2 y = 0
\]

\[
\frac{\partial f}{\partial y} = xe^{-(x^2+y^2)} - 2xy^2 e^{-(x^2+y^2)} = 0 \quad \Rightarrow \quad x - 2xy^2 = 0
\]

\[y \left(1 - 2x^2 \right) = 0 \]
\[x \left(1 - 2y^2 \right) = 0\]

- If \( x = 0 \), then \( y = 0 \) \( \rightarrow \) \((0,0)\)
- If \( y = 0 \), then \( x = 0 \) \( \rightarrow \) \((0,0)\).
- Otherwise, \( x, y = \pm \sqrt{\frac{1}{2}} \)

\[\rightarrow \left( \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right), \left( \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right), \left( -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right), \left( -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right)\]

2. (3 points) Let \( f(x, y) \) be a function of two variables and suppose \((a, b)\) is a critical point of \( f \). Furthermore, suppose:

(a) \( f_{xx}(a, b) = 1 \)
(b) \( f_{xy}(a, b) = 2 \)
(c) \( f_{yy}(a, b) = 7 \)

Is \( f(a, b) \) a local min., local max., or neither? Explain your answer.

\[
\begin{vmatrix}
  f_{xx} & f_{xy} \\
  f_{xy} & f_{yy}
\end{vmatrix} = \begin{vmatrix}
  1 & 2 \\
  2 & 7
\end{vmatrix} = 7 - 4 = 3 > 0
\]

Also, \( f_{xx} > 0 \) at \((a, b)\), so we are curved upwards, meaning this is a \underline{local minimum}.\]
3. (4 points) Suppose you live on the surface \( z^2 = x^2 + y^2 \). The surface is heated by energy from the Sun and the temperature at any given point \((x, y, z)\) is \( T(x, y, z) = x^2 y - z \). Use the Lagrange multiplier method to find the location on the surface with the lowest temperature. (Make sure you don’t divide by zero at some point in your calculations!). You needn’t explain why this is a minimum, though it is something nice to ponder.

\[
\begin{align*}
\text{Constraint} & \\
9(x, y, z) = 2z^2 - x^2 - y^2 = 0
\end{align*}
\]

\[
\begin{align*}
\lambda \nabla g &= \nabla T \\
\rightarrow 2x - 2\lambda x - 2y - 2z &= \langle 2xy, x^2, -1 \rangle \\
\downarrow & \\
\text{we get the system:} & \\
\text{(i) } -2\lambda x &= 2xy \\
\text{(ii) } -2\lambda y &= x^2 \\
\text{(iii) } 2\lambda z &= -1
\end{align*}
\]

\[
\begin{cases}
\lambda = 0 & \rightarrow -1 = 0 \quad \text{impossible} \\
x = 0 & \rightarrow -2\lambda y = 0 \quad \gamma = 0 \quad z = 0 \quad \text{(from constraint)} \\
-1 = 0 & \quad \text{impossible.} \\
\text{(same thing if we start with } y = 0) \quad & \\
\text{Nothing is zero}
\end{cases}
\]

- Now \( \lambda = -\frac{1}{2z} \) and \( y = -\lambda \), meaning \( \gamma = \frac{1}{2z} \). Plugging these into equation "(ii)" yields \( x^2 = \frac{1}{2z^2} \).
- Plugging into the constraint equation, we get:

\[
\begin{align*}
z^2 - \frac{1}{2z^2} - \frac{1}{4z^2} &= 0 \\
\rightarrow z^4 &= \frac{3}{4} \\
\rightarrow z &= \pm \frac{\sqrt{3}}{2} \quad y = \pm \frac{\sqrt{3}}{3} \\
\text{with } \lambda = \frac{1}{2z} \\
x^2 &= \frac{1}{2z^2} = \frac{1}{\sqrt{3}} \quad z (\frac{3}{4})^{\frac{1}{2}} = \frac{1}{\sqrt{3}}
\end{align*}
\]

- Thus, \( F(x, y, z) \) at one of these points is:

\[
+ \frac{1}{\sqrt{3}} \left( \frac{4\sqrt{3}}{3^2} \right) + \frac{4\sqrt{3}}{4} \}
\]

The minimum of these two values is \( \frac{1}{2\sqrt{3}} \left( \frac{4\sqrt{3}}{3} \right) - \frac{4\sqrt{3}}{4} \).