• No calculators allowed.
• Show sufficient work to justify each answer.
• You have 15 minutes for this quiz.

1. (5 points) The function

\[ f(x, y) = x^4 + 4xy + xy^2 \]

has 3 critical points. Calculate the three critical points and indicate (with justification) whether each determines a local maximum value of \( f(x, y) \), a local minimum value of \( f(x, y) \), or a saddle point of \( f(x, y) \).

\[
\begin{align*}
\frac{f_x}{x} &= 4x^3 + 4y + y^2 \\
\frac{f_y}{y} &= 4x + 2xy = 2x(2+y)
\end{align*}
\]

i) \( x = 0 \) , then \( 4y + y^2 = y(4+y) = 0 \)
Then either \( y = 0 \) or \( y = -4 \).

ii) \( y = -2 \) , then \( 4x^3 - 4 = 0 \) \( \Rightarrow x = 1 \).

Thus \( (0, 0) \), \( (0, -4) \), \( (1, -2) \) are critical points

\[
\begin{align*}
\frac{f_{xx}}{xx} &= 12x^2 \\
\frac{f_{yy}}{yy} &= 2x \\
\frac{f_{xy}}{xy} &= 4 + 2y \\
D(0, 0) &= -16 < 0 \\
D(0, -4) &= -16 < 0 \\
D(1, -2) &= 24 > 0, \quad f_{xx}(1, -2) > 0
\end{align*}
\]

Hence \( (0, 0) \) and \( (0, -4) \) are saddle points and \( (1, -2) \) is a local minimum.
2. (5 points) Find the points on the cone \( z^2 = x^2 + y^2 \) that are closest to the point \((4, 2, 0)\).

Let \( g(x, y, z) = x^2 + y^2 - z^2 \)

\[ f(x, y, z) = (x-4)^2 + (y-2)^2 + z^2 \]

\[ \nabla g = (2x, 2y, -2z) \]

\[ \nabla f = (2(x-4), 2(y-2), 2z) \]

Then \( \nabla f = \lambda \nabla g \) for some \( \lambda \in \mathbb{R} \).

i) If \( z = 0 \), since \( z^2 = x^2 + y^2 \), \( x = y = 0 \).

ii) If \( z > 0 \), since \( 2z = -2 \lambda z \), \( \lambda = -1 \).

Then \( 2(x-4) = -2x \) and \( 2(y-2) = -2y \)

so, \( x = 2 \) and \( y = 1 \).

Since \( z^2 = x^2 + y^2 \), \( z = \pm \sqrt{5} \).

Thus \((0, 0, 0)\) and \((2, 1, \pm \sqrt{5})\) are possible candidates.

\[ |(4, 2, 0) - (0, 0, 0)| = \sqrt{20} \]

\[ |(2, 1, \pm \sqrt{5}) - (4, 2, 0)| = \sqrt{10} \]

Hence \((2, 1, \pm \sqrt{5})\) are the points.