Math 241  
Quiz 12 (take-home)  
Spring 2012

Name: Solutions

(circle your TA discussion section)

- DD1, TR 9:00-9:50, Argen West  
- DD3, TR 4:00-4:50, Bo Gwang Jeon  
- DD5, TR 2:00-2:50, Bo Gwang Jeon  
- DD7, TR 8:00-8:50, Daniel Hockensmith  
- DD2, TR 10:00-10:50, Daniel Hockensmith  
- DD4, TR 1:00-1:50, Argen West  
- DD6, TR 4:00-4:50, Euijin Hong  
- DD8, TR 11:00-11:50, Euijin Hong

- You may work with other students in this class. However each student should write up solutions separately and independently – nobody should copy someone else’s work.
- You may use your notes or the textbook.
- Computers are not allowed on any problem. You may use a calculator only for basic arithmetic.
- The quiz should be submitted to Mr. Murphy at the beginning of lecture on Wednesday, May 2nd. Late quizzes will not be accepted.
- There is a higher expectation for the quality of your work on a take-home quiz. Everything should be written logically and legibly with sufficient work to justify each answer. Blank copies of the quiz are available on the course home page.
- Be sure that the pages are nicely stapled – do not just fold the corners.
- Note to TAs and Tutors – you should not help students with these specific problems or go over solutions until after 2pm Wednesday.
1. (3 points) Suppose \( \vec{F} = (e^{xy}, e^{xz}, z^2) \) and \( S \) is the half of the ellipsoid \( 4x^2 + y^2 + 4z^2 = 4 \) that lies to the right of the \( xz \)-plane, oriented in the direction of the positive \( y \)-axis. Use Stokes' Theorem to evaluate \( \iint_S \text{curl} \vec{F} \cdot d\vec{S} \).

\[ \vec{F} = \langle e^{xy}, e^{xz}, z^2 \rangle \Rightarrow \frac{d\vec{F}}{dt} = \langle -ysin(t), 0, 2z \rangle \]

\[ \vec{c} = \langle cos(t), 0, -sin(t) \rangle \]

\[ \vec{F} \cdot \frac{d\vec{c}}{dt} = -sin(t) + cos^3(t)sin(t) \]

By Stokes' Theorem, \( \iint_S \text{curl} \vec{F} \cdot d\vec{S} = \oint_C \vec{F} \cdot d\vec{r} \)

\[ \oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} (-sin(t) + cos^3(t)sin(t)) dt \]

\[ = \left[ cos(t) - \frac{1}{4} cos^4(t) \right]_{2\pi}^{0} \]

\[ = [cos(0) - \frac{1}{4} cos^4(0)] - [cos(0) - \frac{1}{4} cos^4(0)] \]

\[ = [0] - [0] = 0 \]
2. (3 points) Suppose \( \mathbf{F} = (1, x + y, xy - \sqrt{z}) \) and \( C \) is the boundary of the part of the plane \( 3x + 2y + z = 1 \) in the first octant. Given that \( C \) is oriented counterclockwise as viewed from above, use Stokes' Theorem to evaluate \( \int_C \mathbf{F} \cdot d\mathbf{r} \).

\[
\mathbf{F} = \langle x, y, 1 - 3x - 2y \rangle \\
\mathbf{F}_x = \langle 1, 0, -3 \rangle \\
\mathbf{F}_y = \langle 0, 1, -2 \rangle \\
\mathbf{F}_x \times \mathbf{F}_y = \langle 3, 2, 1 \rangle
\]

\[
\text{curl } \mathbf{F} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
x + yz & xy - \sqrt{z} & 1
\end{vmatrix} = \langle -x - y, 1, x + y - \sqrt{z} \rangle
\]

By Stokes' Theorem,
\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_D \text{curl } \mathbf{F} \cdot \mathbf{n} \, dA
\]

\[
= \int_0^1 \int_0^{1/3} (3x - 5y + 1) \, dy \, dx \\
= \int_0^1 \left[ 3xy - \frac{5}{2}y^2 + y \right]_{y=0}^{y=1/3} \\
= \int_0^1 \left( 3x \left( \frac{1}{3} - \frac{5}{2} \cdot \frac{1}{9} \right) - \frac{5}{2} \left( \frac{1}{3} - \frac{5}{2} \cdot \frac{1}{9} \right)^2 + \frac{1}{3} - \frac{5}{6}x \right) \\
= \int_0^1 \left( -\frac{81}{8}x^2 + \frac{15}{2}x - \frac{1}{8} \right) \\
= \left[ -\frac{27}{8}x^3 + \frac{15}{8}x^2 - \frac{1}{8}x \right]_{x=0}^{x=1} = -\frac{1}{8} + \frac{5}{24} - \frac{1}{24} = \frac{1}{24}
\]
3. (4 points) Suppose \( \vec{F} = (x^3 + y^3, y^3 + z^3, z^3 + x^3) \) and \( S \) is the sphere of radius 2 centered at the origin. Use the Divergence Theorem to calculate the flux of \( \vec{F} \) across \( S \).

\[
\text{div} (\vec{F}) = \frac{\partial}{\partial x} (x^3 + y^3) + \frac{\partial}{\partial y} (y^3 + z^3) + \frac{\partial}{\partial z} (z^3 + x^3)
\]
\[= 3x^2 + 3y^2 + 3z^2
\]
\[= 3(x^2 + y^2 + z^2)
\]

By the divergence theorem,

\[
\text{Flux of } \vec{F} \text{ across } S = \iiint_V \text{div} \vec{F} \, dV
\]

\[
= \iiint_{V} 3(x^2 + y^2 + z^2) \, dV
\]

Using spherical coordinates,

\[
= \int_0^{2\pi} \int_0^{\pi} \int_0^2 3 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta
\]

\[
= \left[ \phi \right]_0^{2\pi} \left[ -\cos \phi \right]_0^\pi \left[ \frac{3}{2} \rho^3 \right]_0^2
\]

\[
= 2\pi \cdot 2 \cdot \frac{9}{2}
\]

\[
= \frac{384\pi}{5}
\]