The equations for planes $P_1$, $P_2$ and $P_3$ are

\[ P_1 : 3x - 2y + z = 7 \quad P_2 : x - 4y + 5z = 6 \quad P_3 : 3(x - 5) - 2(y - 6) + (z - 7) = 0 \]

The equations for lines $L_1$, $L_2$, $L_3$, $L_4$ and $L_5$ are

\[
\begin{align*}
L_1 : & \begin{cases} 
    x = 4 + t \\
    y = 3 + 2t \\
    z = -2 + 3t
\end{cases} \\
L_2 : & \begin{cases} 
    x = -3 - 3t \\
    y = 2 + 2t \\
    z = 1 - t
\end{cases} \\
L_3 : & \begin{cases} 
    x = 5 - t \\
    y = 4 - t \\
    z = 3 + 2t
\end{cases} \\
L_4 : & \begin{cases} 
    x = 2 + 2t \\
    y = 5 - 2t \\
    z = 8 - 3t
\end{cases} \\
L_5 : & \begin{cases} 
    x = 5 + t \\
    y = 6 + 2t \\
    z = 7 + 3t
\end{cases}
\]

1. Where does $L_1$ intersect $P_1$?

Plug the values for $x$, $y$, and $z$ (taken from the parametric equations for $L_1$) into the equation of $P_1$. This will give the value of $t$ at the point of intersection. Then, plug this value of $t$ into the parametric equations for $L_1$ to find the coordinates of the point.

\[
3(4 + t) - 2(3 + 2t) + (-2 + 3t) = 7 \quad 12 + 3t - 6 - 4t - 2 + 3t = 7 \quad 2t = 3 \quad t = \frac{3}{2}
\]

\[
x = 4 + \frac{3}{2} = \frac{11}{2}, \quad y = 3 + 2\left(\frac{3}{2}\right) = 6, \quad z = -2 + 3\left(\frac{3}{2}\right) = \frac{5}{2} \quad (x, y, z) = \left(\frac{11}{2}, 6, \frac{5}{2}\right)
\]

2. Do $L_1$ and $L_2$ intersect? If so, where?

At a point of intersection, the lines should have the same $x$-coordinate, the same $y$-coordinate and the same $z$-coordinate. This gives the system of equations $4 + t_1 = -3 - 3t_2$, $3 + 2t_1 = 2 + 2t_2$ and $-2 + 3t_1 = 1 - t_2$. We needed to use $t_1$ and $t_2$ for the parameter $t$ since even when lines intersect they may not reach that point of intersection at the same time. Solving the first two equations in this system gives $t_1 = -17/8$ and $t_2 = -13/8$. Plugging $t_1 = -17/8$ into the equation for $L_1$ gives a $z$-coordinate of $-67/8$, but plugging $t_2 = -13/8$ into the equation for $L_2$ gives a $z$-coordinate of $21/8$. Since the $z$-coordinates are different at the only point for which both the $x$-coordinates and the $y$ coordinates agree, we conclude that the lines do not intersect.
3. What is the angle between $P_1$ and $P_2$?

The angle between $P_1$ and $P_2$ is the same as the angle between their normal vectors $n_1 = 3i - 2j + k$ and $n_2 = i - 4j + 5k$.

$$\cos(\theta) = \frac{n_1 \cdot n_2}{|n_1||n_2|} = \frac{3 + 8 + 5}{\sqrt{9 + 4 + 1} \sqrt{1 + 16 + 25}} = \frac{16}{14\sqrt{3}} = \frac{8}{7\sqrt{3}} \quad \theta = \cos^{-1} \left( \frac{8}{7\sqrt{3}} \right)$$

There are actually two angles. The other one is the supplementary angle $\pi - \cos^{-1} \left( \frac{8}{7\sqrt{3}} \right)$.

4. Find the line through the points $(5, 6, 7)$ and $(7, -2, 7)$.

The displacement vector joining the two points is

$$v = (7 - 5)i + ((-2) - 6)j + (7 - 7)k = 2i - 8j$$

This is a direction vector for the line. One possible set of parametric equations for the line through the two points is

$$L : \begin{cases} x = 5 + 2t \\ y = 6 - 8t \\ z = 7 \end{cases}$$

5. Find the line through the point $(5, 6, 7)$ that is perpendicular to $P_1$.

A normal vector for $P_1$ is a direction vector for the desired line, so $d_1 = n_1 = 3i - 2j + k$. One possible set of parametric equations for the line is

$$L : \begin{cases} x = 5 + 3t \\ y = 6 - 2t \\ z = 7 + t \end{cases}$$

6. Find the angle between $L_1$ and $P_1$.

Let $\theta$ be the angle between $L_1$ and $P_1$, let $d_1$ be a direction vector for $L_1$, and let $n_1$ be a normal vector for $P_1$. Thus, $d_1 = i + 2j + 3k$ and $n_1 = 3i - 2j + k$. Let $\alpha$ be the angle between these vectors.

$$\cos(\alpha) = \frac{|d_1 \cdot n_1|}{|d_1||n_1|} = \frac{|3 - 4 + 3|}{\sqrt{1 + 4 + 9} \sqrt{9 + 4 + 1}} = \frac{2}{14} = \frac{1}{7} \quad \alpha = \cos^{-1} \left( \frac{1}{7} \right)$$

$$\theta = \frac{\pi}{2} - \alpha = \frac{\pi}{2} - \cos^{-1} \left( \frac{1}{7} \right)$$

7. Find the plane through the point $(5, 6, 7)$ that is parallel to $P_1$.

The normal vector for $P_1$, $n_1 = 3i - 2j + k$, is also a normal vector for the desired plane. One way to write the equation for the plane is $3(x - 5) - 2(y - 6) + (z - 7) = 0$. 
8. Find the plane through the points \((0, 0, 0)\), \((5, 6, 7)\), and \((7, -2, 7)\).

We construct the displacement vector \(\mathbf{v} = 5\mathbf{i} + 6\mathbf{j} + 7\mathbf{k}\) joining \((0, 0, 0)\) to \((5, 6, 7)\) and the displacement vector \(\mathbf{w} = 7\mathbf{i} - 2\mathbf{j} + 7\mathbf{k}\) joining \((0, 0, 0)\) to \((7, -2, 7)\). The vector \(\mathbf{n} = \mathbf{v} \times \mathbf{w}\) is perpendicular to the plane containing the three points.

\[
\mathbf{n} = \mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\
5 & 6 & 7 \\
7 & -2 & 7 \end{vmatrix} = \mathbf{i}(42 - (-14)) - \mathbf{j}(35 - 49) + \mathbf{k}(-10 - 42) = 56\mathbf{i} + 14\mathbf{j} - 52\mathbf{k}
\]

Since \((0, 0, 0)\) is in the plane, one form for the equation of the plane is

\[56(x - 0) + 14(y - 0) - 52(z - 0) = 0\]

That is, \(56x + 14y - 52z = 0\).

9. Find the plane determined by the intersecting lines \(L_3\) and \(L_4\).

We first verify that the lines intersect by solving the system of equations \(5 - t_1 = 2 + 2t_2\), \(4 - t_1 = 5 - 2t_2\) and \(3 + 2t_1 = 8 - 3t_2\). See problem 2 for an explanation of why we use \(t_1\) and \(t_2\) here. This system has a solution at \(t_1 = 1\) and \(t_2 = 1\), so the lines do intersect and we see that \((4, 3, 5)\) is the point of intersection.

The point \((4, 3, 5)\) is in the plane containing the lines. A vector perpendicular to the plane containing the lines is \(\mathbf{n} = \mathbf{d}_3 \times \mathbf{d}_4\) where \(\mathbf{d}_3 = -\mathbf{i} - \mathbf{j} + 2\mathbf{k}\) is a direction vector for \(L_3\) and \(\mathbf{d}_4 = 2\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}\) is a direction vector for \(L_4\).

\[
\mathbf{n} = \mathbf{d}_3 \times \mathbf{d}_4 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\
-1 & -1 & 2 \\
2 & -2 & -3 \end{vmatrix} = \mathbf{i}(3 - (-4)) - \mathbf{j}(3 - 4) + \mathbf{k}(2 - (-2)) = 7\mathbf{i} + \mathbf{j} + 4\mathbf{k}
\]

One form for the equation of the plane is \(7(x - 4) + (y - 3) + 4(z - 5) = 0\).

10. Find the plane through the point \((5, 6, 7)\) that is perpendicular to \(L_1\).

Since \(L_1\) is perpendicular to the plane, the direction vector for \(L_1\) is a normal vector for the plane. That is, \(\mathbf{n} = \mathbf{d}_1 = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}\). One form for the equation of the plane is \((x - 5) + 2(y - 6) + 3(z - 7) = 0\).
11. Find the distance from the point \((5, 6, 7)\) to \(L_1\).

Note that \((4, 3, -2)\) is a point on \(L_1\) and that \(d_1 = i + 2j + 3k\) is a direction vector for \(L_1\). We construct the displacement vector \(v = i + 3j + 9k\) joining \((4, 3, -2)\) to \((5, 6, 7)\).

From the diagram we see that \(\sin \theta = \frac{D}{|v|}\) which gives us the distance

\[
D = |v| \sin(\theta) = \frac{|d_1||v| \sin(\theta)}{|d_1|} = \frac{|d_1 \times v|}{|d_1|}
\]

\[
|d_1| = \sqrt{1 + 4 + 9} = \sqrt{14}
\]

\[
d_1 \times v = \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 1 & 3 & 9 \end{vmatrix} = i(18 - 9) - j(9 - 3) + k(3 - 2) = 9i - 6j + k
\]

\[
|d_1 \times v| = \sqrt{81 + 36 + 1} = \sqrt{118}
\]

The distance is

\[
D = \frac{|d_1 \times v|}{|d_1|} = \frac{\sqrt{118}}{\sqrt{14}} = \sqrt{\frac{59}{7}}.
\]

12. Find the distance from the point \((5, 6, 7)\) to \(P_1\).

Note that \((0, 0, 7)\) is a point on \(P_1\) and that \(n_1 = 3i - 2j + k\) is a normal vector for \(P_1\). We construct the displacement vector \(v = -5i - 6j\) joining \((5, 6, 7)\) to \((0, 0, 7)\).

From the diagram we see that \(|\cos \theta| = \frac{D}{|v|}\) which gives us the distance

\[
D = |v||\cos(\theta)| = \frac{|n_1||v||\cos(\theta)|}{|n_1|} = \frac{|n_1 \cdot v|}{|n_1|} = \frac{|-15 + 12 + 0|}{\sqrt{9 + 4 + 1}} = \frac{3}{\sqrt{14}}
\]
13. Find the distance between the parallel planes \( P_1 \) and \( P_3 \).

Once we recognize that \((5, 6, 7)\) is on plane \( P_3 \), this becomes identical to problem 12.

14. Find the distance between the parallel lines \( L_1 \) and \( L_5 \).

Once we recognize that \((5, 6, 7)\) is on the line \( L_5 \), this becomes identical to problem 11.

15. Find the line of intersection of \( P_1 \) and \( P_2 \).

The line \( L \) is perpendicular to both \( n_1 \) and \( n_2 \) where \( n_1 = 3i - 2j + k \) is a normal vector for \( P_1 \) and \( n_2 = i - 4j + 5k \) is a normal vector for \( P_2 \). Since \( n_1 \times n_2 \) is also perpendicular to both \( n_1 \) and \( n_2 \), we see that \( d = n_1 \times n_2 \) is a direction vector for \( L \).

In addition to finding this direction vector, we need a point on the line. That is, we need any point that is on both \( P_1 \) and \( P_2 \). We may let \( z = 0 \). This makes the equations for the planes \( 3x - 2y = 7 \) and \( x - 4y = 6 \). We can solve this system of equations to get \( x = 16/10 \) and \( y = -11/10 \), so \((1.6, -1.1, 0)\) is on both planes (and thus on the line).

\[
d = n_1 \times n_2 = \begin{vmatrix} i & j & k \\ 3 & -2 & 1 \\ 1 & -4 & 5 \end{vmatrix} = i(-10 - (-4)) - j(15 - 1) + k(-12 - (-2)) = -6i - 14j - 10k
\]

One possible set of parametric equations for the line is

\[
L : \begin{cases} x = 1.6 - 6t \\ y = -1.1 - 14t \\ z = -10t \end{cases}
\]