Name ________________

- You have 15 minutes
- No calculators
- Show sufficient work

1. (2 points each) State whether each series converges or diverges. You must explain your reasoning. If a series converges, then you should also compute its sum.

   (a) \( \sum_{k=0}^{\infty} \frac{2^{2k}}{3^{k+1}} = \frac{1}{3} + \frac{4}{9} + \frac{16}{27} + \frac{64}{81} + \cdots \)

   Geometric series, ratio = \( \frac{4}{3} \)

   Since \( \frac{4}{3} > 1 \), the series diverges.

   Could also use divergence test since

   \( \lim_{k \to \infty} (\frac{2^{2k}}{(3^{k+1})}) = \infty \) (not 0)

   (b) \( \sum_{k=0}^{\infty} \frac{3}{5k + 5} = \frac{3}{5} \sum_{k=0}^{\infty} \frac{1}{k+1} \)

   \( = \frac{3}{5} (1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots) \)

   Harmonic series

   This series diverges since the harmonic series diverges.

2. (2 points) For which values of \( x \) does the following series converge? What is the sum of the series for these values of \( x \)?

   \( \sum_{k=0}^{\infty} \left( \frac{2x + 5}{7} \right)^k = 1 + \left( \frac{2x+5}{7} \right) + \left( \frac{2x+5}{7} \right)^2 + \cdots \)

   Geometric series, ratio = \( \frac{2x+5}{7} \)

   Series converges if and only if \( \left| \frac{2x+5}{7} \right| < 1 \)

   \( -1 < \frac{2x+5}{7} < 1 \) \( \iff \) \( -7 < 2x + 5 < 7 \) \( \iff \) \( -12 < 2x < 2 \), \( -6 < x < 1 \)

   Series converges to

   \[ \frac{1}{1 - \left( \frac{2x+5}{7} \right)} = \frac{7}{2 - 2x} \]
3. (2 points) Prove that the given sequence is decreasing. This requires more than just listing the first few terms.

\[ \frac{a_{n+1}}{a_n} = \frac{2^{n+1}/(n+1)!}{2^n/n!} = \frac{2^{n+1} \cdot n!}{2^n \cdot (n+1)!} = \frac{2 \cdot (n+1)}{n+1} \leq 1 \quad \text{for } n \geq 2 \Rightarrow \text{sequence is decreasing} \]

\[ a_{n+1} - a_n < 0 \Rightarrow a_{n+1} < a_n \]

4. (2 points) Does the following series converge or diverge? If the series converges, find its exact sum. It will help to rewrite the sum using sigma notation \( \Sigma \).

\[
\frac{2}{1 \cdot 3} + \frac{2}{2 \cdot 4} + \frac{2}{3 \cdot 5} + \frac{2}{4 \cdot 6} + \frac{2}{5 \cdot 7} + \frac{2}{6 \cdot 8} + \ldots
\]

\[
= \sum_{k=1}^{\infty} \frac{2}{k(k+2)} = \lim_{n \to \infty} \sum_{k=1}^{n} \frac{2}{k(k+2)} = \lim_{n \to \infty} \sum_{k=1}^{n} \left( \frac{1}{k} - \frac{1}{k+2} \right)
\]

\[
= \lim_{n \to \infty} \left( 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2} \right) = 1 + \frac{1}{2} - 0 - 0 = \frac{3}{2} \text{ (converges)}
\]
Note: To simply show convergence without computing the sum we see
\[
\frac{2}{1 \cdot 3} + \frac{2}{2 \cdot 4} + \frac{2}{3 \cdot 5} + \cdots
\]
\[
< \frac{2}{1 \cdot 1} + \frac{2}{2 \cdot 2} + \frac{2}{3 \cdot 3} + \cdots
\]
\[
= \sum_{k=1}^{\infty} \frac{2}{k^2}
\]
\[
= 2 \cdot \sum_{k=1}^{\infty} \frac{1}{k^2}
\]

This is twice a convergent $p$-series ($p=2>1$) so it converges. Since our original series has positive terms, we see upon comparison that it converges.