MATH 231  Test 2  Spring 2011

Name ________________________________

(circle your TA discussion section)

- CD1, TR 9:00-9:50, Vicki Reuter
- CD3, TR 12:00-12:50, Ilkyoo Choi
- CD5, TR 9:00-10:50, Paul Spiegelhalter
- CD7, TR 12:00-12:50, Nathan Orlow
- CD9, TR 3:00-3:50, Grace Work

- ED1, TR 2:00-2:50, Yat Sen Wong
- ED3, TR 4:00-4:50, Yat Sen Wong
- ED5, TR 12:00-12:50, Bolor Turmunkh
- ED7, TR 4:00-4:50, Hannah Kolb

- CD2, TR 10:00-10:50, Vicki Reuter
- CD4, TR 2:00-2:50, Tom Mahoney
- CD6, TR 1:00-2:50, Kelly Funk
- CD8, TR 1:00-1:50, Nathan Orlow

- ED2, TR 3:00-3:50, Bolor Turmunkh
- ED4, TR 9:00-9:50, Brian Collier
- ED6, TR 3:00-3:50, Hannah Kolb

- Sit in your assigned seat (shown below).
- Do not open this test booklet until I say START.
- Turn off all electronic devices and put away all items except a pen/pencil and an eraser.
- Unless otherwise stated, you must show sufficient work to justify each answer.
- While the test is in progress, we will not answer questions concerning the test material.
- Quit working and close this test booklet when I say STOP.
- Quickly turn in your test to me or a TA and show your Student ID.

FRONT OF ROOM - 314 Altgeld Hall
1. (6 points) Given that \( E_T \) is the error in using the Trapezoidal Rule with \( n \) subintervals of equal width to approximate \( \int_a^b f(x) \, dx \), fill in the blanks appropriately to show the book's information concerning this error term.

If \( \left| f''(x) \right| \leq K \) for \( a \leq x \leq b \), then \( |E_T| \leq \frac{K(b-a)^3}{12n^2} \).

2. (7 points) Use Simpson's Rule with 4 subintervals of equal width to approximate \( \int_0^8 f(x) \, dx \) given that the graph of \( y = f(x) \) goes through points \((0,3), (2,3), (4,6), (6,4)\) and \((8,14)\). Show the calculations you used to obtain your answer.

\[
S_4 := \frac{\Delta x}{3} \left( f(0) + 4f(2) + 2f(4) + 4f(6) + f(8) \right)
\]

\[
= \frac{2}{3} \left( 3 + 4 \cdot 3 + 2 \cdot 6 + 4 \cdot 4 + 14 \right)
\]

\[
= \frac{2}{3} \left( 3 + 12 + 12 + 16 + 14 \right)
\]

\[
= \frac{2}{3} \left( 47 \right)
\]

\[
= 3 \cdot 8
\]

\[
= 28
\]
3. (10 points) A surface is generated when the curve \( y = (x+3)^{1/2} \) with \( 13 \leq x \leq 33 \) is revolved around the \( y \)-axis.

\[
d\frac{y}{dx} = \frac{1}{2} (x+3)^{-1/2}
\]

(a) Set up, but do not evaluate, an integral with respect to \( x \) which represents the surface area.

\[
\int_{13}^{33} 2\pi x \sqrt{1 + \left(\frac{1}{2}(x+3)^{-1/2}\right)^2} \, dx
\]

(b) Set up, but do not evaluate, an integral with respect to \( y \) which represents the surface area. The limits of integration should be different in parts (a) and (b).

\[
\int_{4}^{6} 2\pi (y^2 - 3) \sqrt{1 + (\frac{1}{2}y^{-1})^2} \, dy
\]

\( y(13) = 4 \)

\( y(33) = 6 \)
4. (12 points) For which values of $x$ does the following series converge? Your answer should be written in the form $a < x < b$ or $a \leq x \leq b$ for some values of $a$ and $b$. You do not need to find the sum of the series.

$$1 + \frac{2x}{3} + \frac{4x^2}{9} + \frac{8x^3}{27} + \frac{16x^4}{81} + \frac{32x^5}{243} + \frac{64x^6}{729} + \cdots$$

**Geom.**  \[ r = \frac{2x}{3} \]

**Conv if**  \[ \left| \frac{2x}{3} \right| < 1 \]

\[ -1 < \frac{2x}{3} < 1 \]

\[ -3 < 2x < 3 \]

\[ -\frac{3}{2} < x < \frac{3}{2} \]

5. (21 points) Circle true if the given statement is always true. Otherwise circle false.

(a) (true or false) If the sequence $\{a_k\}_{k=1}^{\infty}$ converges to a number $p$ with $p > 1$, then the series $\sum_{k=1}^{\infty} a_k$ converges.

**Series div when** \( \lim_{k \to \infty} a_k \neq 0 \)

(b) (true or false) The series $\sum_{k=8}^{\infty} \frac{1}{3k}$ diverges.

\[ = \frac{1}{3} (\text{tail end of harmonic series}) \]

(c) (true or false) The series $\sum_{k=1}^{\infty} \frac{1}{k \sqrt{k}}$ diverges.

\[ = \sum_{k=1}^{\infty} \frac{1}{k^{3/2}} \]

\( p - \text{series with } \)

\( p = 3/2 > 1 \) **conv**
(d) (true or false) If \( \lim_{k \to \infty} a_k = 0 \), then the infinite series \( \sum_{k=1}^{\infty} a_k \) converges.

\[
\lim_{k \to \infty} \frac{1}{k^2} = 0 \text{ but } \sum_{k=1}^{\infty} \frac{1}{k} \text{ diverges.}
\]

(e) (true or false) If the series \( \sum_{k=1}^{\infty} a_k \) converges, then the sequence \( \{a_k\}_{k=1}^{\infty} \) converges.

\[
\lim_{k \to \infty} a_k = 0
\]

(f) (true or false) If the series \( \sum_{k=1}^{\infty} a_k \) diverges, then the sequence \( \{a_k\}_{k=1}^{\infty} \) diverges.

\[
\sum_{k=1}^{\infty} \frac{1}{k^2} \text{ diverges but } \lim_{k \to \infty} \frac{1}{k} = 0
\]

(g) (true or false) If the sequence \( \{a_k\}_{k=1}^{\infty} \) is decreasing and \( a_k \geq \frac{1}{k} \) for all positive integers \( k \), then the sequence \( \{a_k\}_{k=1}^{\infty} \) converges.

\[
\sum_{k=1}^{\infty} \frac{1}{k^2} \text{ decreases & \{a_k\} is bounded below by 0}
\]

6. (8 points) The following alternating series converges to a sum \( S \).

\[
S = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{8k} = \frac{1}{8} - \frac{1}{16} + \frac{1}{24} - \frac{1}{32} + \ldots
\]

How many of the beginning terms of the infinite series should you add together to get an estimate for its sum \( S \) which is correct to 2 decimal places?

\[
S_n = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{8k}
\]

\[
\text{Error} \leq \frac{1}{8(n+1)} = 0.005
\]

\[
\frac{1}{8(n+1)} \leq \frac{1}{200}
\]

\[
8(n+1) \geq 200
\]

\[
k+1 \geq 25
\]

\[
k \geq 24
\]
7. (8 points) Determine the sum of the convergent series \( \sum_{k=0}^{\infty} \frac{48}{2^{2k+1}} \). Simplify your answer.

\[
\frac{48}{2} + \frac{48}{2^3} + \frac{48}{2^5} + \cdots
\]

\( \text{Geom} \) \( |r| = \frac{1}{4} < 1 \) \( \text{converges} \)

\[
\text{Sum} = \frac{24}{1 - \frac{1}{4}} = \frac{24}{\frac{3}{4}} = 24 \cdot \frac{4}{3} = \boxed{32}
\]

8. (8 points) Determine the sum of the convergent series \( \sum_{k=1}^{\infty} \left( \frac{k}{k+3} - \frac{k+1}{k+4} \right) \). Simplify your answer.

\[
\lim_{n \to \infty}
\begin{align*}
\frac{1}{4} - \frac{3}{5} \\
\frac{3}{5} - \frac{3}{6} \\
\vdots
\end{align*}
\]

\[
\lim_{n \to \infty} \left( \frac{n-1}{n+2} - \frac{n}{n+3} \right) = \frac{1}{4} - 1 = \boxed{-\frac{3}{4}}
\]
9. (10 points) Determine whether the given series converges or diverges. Include a carefully written proof to justify your claim. Your proof should state any convergence or divergence tests you are using, why the test is applicable to this series, and why you can conclude that the series converges or diverges.

\[ \sum_{n=1}^{\infty} \frac{n^2 + 5}{\sqrt{n^6 + 10}} \text{ behaves like } \sum \frac{n^2}{\sqrt{n^6}} = \sum \frac{1}{n} \]

so we expect divergence.

**Proof**

\[ \frac{n^2 + 5}{\sqrt{n^6 + 10}} \geq \frac{n^2}{\sqrt{n^6 + 10}n^6} = \frac{1}{n}, n \geq 0 \]

since \( \sum \frac{1}{n} = \sum \frac{1}{\text{harmonic}} \)

our series \( \sum \frac{n^2 + 5}{\sqrt{n^6 + 10}} \) conv. by comparison

**Alternate Proof**

\[ a_n = \frac{n^2 + 5}{\sqrt{n^6 + 10}} \geq 0, \quad b_n = \frac{1}{n} > 0 \]

\[ \lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{n^2 + 5}{\sqrt{n^6 + 10} \cdot \frac{1}{n^2}} \]

\[ = \lim_{n \to \infty} \frac{n^3 + 5n}{n^3 \sqrt{1 + \frac{10}{n^2}}} \]

\[ = \lim_{n \to \infty} \frac{1 + \frac{5n^2}{n^3}}{\sqrt{1 + \frac{10}{n^2}}} = 1 (>0, <\infty) \]

since \( \sum \frac{1}{n} \text{ div (harmonic)}, \) the series \( \sum \frac{n^2 + 5}{\sqrt{n^6 + 10}} \text{ div by lim, comp test} \)
10. (10 points) Determine whether the given series converges or diverges. Include a carefully written proof to justify your claim. Your proof should state any convergence or divergence tests you are using, why the test is applicable to this series, and why you can conclude that the series converges or diverges.

\[ \sum_{n=2}^{\infty} \frac{\ln(n)}{\sqrt{n}} \geq \frac{1}{\sqrt{n}} \geq 0 \text{ for } n \geq 3 \]

since \( \sum \frac{1}{\sqrt{n}} = \sum \frac{1}{n^{1/2}} \text{ diverges (p-series, } p = \frac{1}{2} \leq 1) \)

Our series \( \sum \frac{\ln(n)}{\sqrt{n}} \text{ diverges by comparison} \)

**Alternate Proof**

\( \frac{\ln x}{\sqrt{x}} \text{ is cont, pos, decr.} \)

For \( x > e^2 \)

proof of decr \( \left( \frac{\ln x}{\sqrt{x}} \right)' = \frac{\frac{x}{\sqrt{x}} - \ln x \cdot \frac{x}{2}\sqrt{x} - \frac{x}{2}}{(\sqrt{x})^2} \)

\[ = 1 - \frac{\ln x}{x \sqrt{x}} (\text{multiplied}) \]

\( < 0 \text{ for } x > e^2 \)

\[ \int_{2}^{\infty} \frac{\ln x}{\sqrt{x}} \, dx = \lim_{b \to \infty} \int_{2}^{b} \frac{\ln x}{\sqrt{x}} \, dx \]

by parts

\[ u = \ln x, \quad dv = x^{-1/2} dx \]

\[ du = \frac{1}{x} dx, \quad v = 2x^{1/2} \]

since integral diverges

so does the series diverges

by integral test

\[ \lim_{b \to \infty} \left( 2 \sqrt{b} \ln b - 2 \sqrt{b} \ln 2 - \left( \frac{b \sqrt{b} - b}{2} \right) \right) = 0 \]

\[ \lim_{b \to \infty} \left( 2 \sqrt{b} (\ln b - 2) - 2 \sqrt{b} + 4 \sqrt{b} \right) = \infty \]
Students – do not write on this page!

1. (6 points) __________________________

2. (7 points) __________________________

3. (10 points) __________________________

4. (12 points) __________________________

5. (21 points) __________________________

6. (8 points) ____________________________

7. (8 points) ____________________________

8. (8 points) ____________________________

9. (10 points) __________________________

10. (10 points) __________________________

TOTAL (100 points) ________________