Solutions

- Sit in your assigned seat (circled below).
- Circle your TA discussion section.
- Do not open this test booklet until I say START.
- Turn off all electronic devices and put away all items except a pen/pencil and an eraser.
- Remove hats and sunglasses.
- You must show sufficient work to justify each answer.
- While the test is in progress, we will not answer questions concerning the test material.
- Do not leave early unless you are at the end of a row.
- Quit working and close this test booklet when I say STOP.
- Quickly turn in your test to me or a TA and show your Student ID.

| AD1, TR 9:00-10:50, Andrew McConvey | ADJ, TR 9:00-9:50, Mi Young Jang |
| AD2, TR 10:00-10:50, Derrek Yager   | ADK, TR 10:00-10:50, Stephen Berning |
| ADA, TR 8:00-8:50, Mi Young Jang    | ADL, TR 11:00-11:50, Adam Wagner |
| ADB, TR 9:00-9:50, Stephen Berning  | ADM, TR 12:00-12:50, Adam Wagner |
| ADC, TR 10:00-10:50, Sarah Yeakel   | ADN, TR 1:00-1:50, Mychael Sanchez |
| ADD, TR 11:00-11:50, Michael Livesay| ADO, TR 2:00-2:50, Mychael Sanchez |
| ADE, TR 12:00-12:50, George Shakan  | ADP, TR 3:00-3:50, Albert Tamazyan |
| ADF, TR 1:00-1:50, Albert Tamazyan  | ADQ, TR 4:00-4:50, George Shakan |
| ADG, TR 2:00-2:50, Alonza Terry     | ADR, TR 9:00-9:50, Michael Livesay |
| ADH, TR 3:00-3:50, Alonza Terry     | |

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FRONT OF ROOM – 100 Materials Science and Engineering Building
1. (5 points) Suppose that a polynomial \( f(x) \) satisfies the following conditions.

- \( f(4) = 8 \)
- \( f'(4) = 3 \)
- \( f''(4) = 6 \)
- \( f'''(4) = 2 \)

Use a linear approximation to estimate the value of \( f(3.8) \). Simplify and write your answer in decimal form.

\[
\text{point: } (4, f(4)) = (4, 8) \\
\text{slope: } f'(4) = 3 \\
\text{tangent line: } y - 8 = 3(x - 4) \Rightarrow y = 8 + 3(x - 4) \\
f(x) \approx 8 + 3(x - 4) \text{ for } x \text{ near 4}
\]

\[
f(3.8) \approx 8 + 3(3.8 - 4) \\
f(3.8) \approx 7.4
\]

2. (5 points) Fill in the missing information to show that the area between the \( x \)-axis and the graph of \( f(x) = \frac{1}{x^2 + 1} \) on the interval \([-2, 2]\) can be expressed as the limit of a right Riemann sum. The only variables appearing in your limit should be \( n \) and \( k \). Do not evaluate this limit.

\[
\text{AREA} = \lim_{n \to \infty} \sum_{k=1}^{n} \left[ \frac{1}{(-2 + k \cdot \Delta x)^2 + 1}, \frac{4}{n} \right]
\]

\[
\Delta x = \frac{2 - (-2)}{n} = \frac{4}{n}
\]

\[
x^*_k = -2 + k \cdot \frac{4}{n}
\]
3. (10 points) Let \( g(x) = \int_{x^5}^{x^5} \sin(t^2) \, dt \). Find \( g'(x) \).

\[
g(x) = -\int_{x^5}^{x^5} \sin(t^2) \, dt
\]

Let \( u = x^5 \), so \( \frac{du}{dx} = 5x^4 \)

Now \( g = -\int_{x^5}^{x^5} \sin(t^2) \, dt \) has derivative

\[
\frac{dg}{du} = -\sin(u^2) \quad \text{by F.T.C. (part 1)}
\]

\[
\frac{dg}{dx} = \frac{dg}{du} \cdot \frac{du}{dx} = -\sin((x^5)^2) \cdot 5x^4
\]

(by the chain rule)

\[
g'(x) = -5x^4 \sin(x^{10})
\]

(can obtain more quickly)

4. (10 points) Some of the values of a polynomial \( f(x) \) are shown in the table. If \( g(x) = 8xf'(x^2) \), then find the average value of \( g(x) \) on the interval \([0, 2]\). Simplify your answer.

<table>
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<td>233</td>
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</tbody>
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\[
g_{\text{ave}} = \frac{1}{2 - 0} \int_{0}^{2} g(x) \, dx
\]

\[
= \frac{1}{2} \int_{0}^{2} 8xf'(x^2) \, dx
\]

\[
= \frac{1}{2} \left[ \frac{8}{2} f'(u) \right]_{0}^{4}
\]

\[
= \frac{1}{2} \left[ f(4) - f(0) \right]_{0}^{4}
\]

\[
= 2 \left[ f(4) - f(0) \right] = 2(36) = 72
\]
5. (10 points) There are currently 400 rabbits living on an island. A biologist uses a model which predicts that the population will increase by \(2t + 5\) rabbits per year where \(t\) represents the number of years from today. According to the biologist’s model, how many rabbits will be living on the island 10 years from now?

\[
\text{Population at } t=10 = \text{current population} + \text{net change in population from } t=0 \text{ to } t=10
\]

\[
= 400 + \int_0^{10} (2t+5) \, dt
\]

\[
= 400 + \left[ t^2 + 5t \right]_0^{10}
\]

\[
= 400 + \left[ (100+50) - 0 \right]
\]

\[
= 550 \text{ rabbits}
\]

6. (10 points) Evaluate the definite integral. Simplify your answer.

\[
\int_1^3 \frac{2x}{x^2 + 1} \, dx = \int_0^{10} \frac{1}{u} \, du = \left[ \ln(u) \right]_2^{10}
\]

\[
u = x^2 + 1
\]

\[du = 2x \, dx\]

\[x=1 \Rightarrow u=1^2+1=2\]

\[x=3 \Rightarrow u=3^2+1=10\]

\[
= \ln(10) - \ln(2)
\]

\[
= \ln\left(\frac{10}{2}\right)
\]

\[
= \ln(5)
\]
7. (10 points) Evaluate the indefinite integral.

\[ \int \frac{\sin^2 x}{\sec x \csc^4 x} \, dx = \int \frac{\sin^2 x}{\cos x \sin^4 x} \, dx \]

\[ = \int \sin^6 x \cos x \, dx \]

\[ = \int u^6 \, du \]

\[ = \frac{1}{7} u^7 + C \]

\[ = \frac{1}{7} \sin^7 x + C \]

8. (10 points) Evaluate the indefinite integral.

\[ \int \cot^6 x \csc^4 x \, dx = \int \cot^6 x \csc^2 x \csc^2 x \, dx \]

\[ = \int \cot^6 x (\cot^2 x + 1) \csc^2 x \, dx \]

\[ = \int u^6 (u^2 + 1) (-du) \]

\[ = \int (-u^{18} - u^{16}) \, du \]

\[ = \frac{1}{19} u^{19} - \frac{1}{17} u^{17} + C \]

\[ = \frac{1}{19} \cot^9 x - \frac{1}{17} \cot^7 x + C \]
9. (10 points) Evaluate the indefinite integral.

\[ \int 4x^7 (x^4 + 1)^{15} \, dx = 5 \cdot \left( x^4 + 1 \right)^{15} x^3 \, dx \]

\[ u = x^4 + 1 \quad \Rightarrow \quad du = 4x^3 \, dx \]

\[ x^4 = u - 1 \]

\[ = 5 \cdot (u - 1)^{15} \, du \]

\[ = \frac{5}{15} u^{17} - \frac{1}{16} u^6 + C \]

\[ = \frac{1}{5} (x^4 + 1)^{17} - \frac{1}{16} (x^4 + 1)^6 + C \]

10. (10 points) Evaluate the indefinite integral.

\[ \int \frac{5x^2 + 1}{x^2 + 1} \, dx = 5 \left( 5 - \frac{4}{x^4 + 1} \right) \, dx \]

\[ = 5 \cdot 5 \, dx - 4 \cdot \frac{1}{x^4 + 1} \, dx \]

\[ = 5x - 4 \arctan(x) + C \]

Polynomial long division:

\[ \frac{5}{x^2 + 1} \]

\[ \frac{5x^3 + 1}{x^2 + 1} \]

\[ = 5 + \frac{-4}{x^4 + 1} \]
11. (10 points) The graphs of $y = 2e^x$ and $y = 18e^{-x}$ are shown along with their point of intersection $(x, y) = (\ln 3, 6)$. Let $R$ be the finite region bounded by these two curves and the $y$-axis. By integrating with respect to $x$, set up, but do not evaluate, definite integrals which represent the given quantities.

(a) The volume of the solid obtained when $R$ is revolved around the vertical line $x = -2$.

$$V = \int_0^{\ln 3} (\text{surface area}) \, dx$$

$$V = \int_0^{\ln 3} 2\pi r \cdot h \, dx$$

$$V = \int_0^{\ln 3} 2\pi (x+2)(18e^{-x}-2e^x) \, dx$$

(b) The volume of the solid obtained when $R$ is revolved around the horizontal line $y = 22$.

$$V = \int_0^{\ln 3} (\text{cross-sectional area}) \, dx$$

$$V = \int_0^{\ln 3} (\pi r^2_{\text{out}} - \pi r^2_{\text{in}}) \, dx$$

$$V = \int_0^{\ln 3} (\pi (12e^x)^2 - \pi (2e^x - 18e^{-x})^2) \, dx$$

$$y = 22$$
1. (5 points) ____________________________

2. (5 points) ____________________________

3. (10 points) ____________________________

4. (10 points) ____________________________

5. (10 points) ____________________________

6. (10 points) ____________________________

7. (10 points) ____________________________

8. (10 points) ____________________________

9. (10 points) ____________________________

10. (10 points) ____________________________

11. (10 points) ____________________________

TOTAL (100 points) ____________________________