Math 220 (section AD?) Quiz 10 Spring 2013

Name SOLUTIONS

- You have 20 minutes
- No calculators
- Show sufficient work

1. (2 points) Let $g(x) = \int_{-4}^{x^2} f(t)\,dt$. Determine the equation for the line tangent to the graph of $g(x)$ at $x = 2$ given the following information about f.

   - f is continuous on the interval $(-\infty, \infty)$
   - f is an odd function
   - $f(4) = 20$

$g(2) = \int_{-4}^{4} f(t)\,dt = 0$ since f is odd.

Thus the point of tangency is $(2, 0)$

$g'(\cancel{x}) = f(x^2)\cdot(x^2)' = f(x^2)\cdot 2x$ (used chain rule — see lect, or disc. notes for details)

$g'(2) = f(4)\cdot 2(2) = f(4)\cdot 4 = 20\cdot 4 = 80$

Thus the slope of the tangent line is 80

tangent line $y - 0 = 80(x-2)$

$y = 80x - 160$

2. ~~(2 points) Precisely state~~ *The Mean Value Theorem.*

Suppose the following:

① f is continuous on $[a, b]$

② f is differentiable on (a, b)

Then $f'(c) = \dfrac{f(b) - f(a)}{b - a}$ for some c in (a, b)

3. (4 points) Evaluate the following integrals. For problem (a) it will be helpful to first write it as the sum of two integrals.

(a) $\displaystyle\int \frac{x^5+x^2}{1+x^6}\,dx = \int \frac{x^5}{1+x^6}\,dx + \int \frac{x^2}{1+x^6}\,dx$

$$= \frac{1}{6}\ln(1+x^6) + \frac{1}{3}\arctan(x^3) + C$$

$$\int \frac{x^5}{1+x^6}\,dx = \frac{1}{6}\int \frac{1}{u}\,du = \frac{1}{6}\ln|u| + C_1 = \frac{1}{6}\ln(1+x^6) + C_1$$

$u = 1+x^6$

$\frac{du}{dx} = 6x^5 \Rightarrow du = 6x^5\,dx \Rightarrow \frac{1}{6}du = x^5\,dx$

$$\int \frac{x^2}{1+x^6}\,dx = \int \frac{x^2}{1+(x^3)^2}\,dx = \frac{1}{3}\int \frac{1}{1+u^2}\,du = \frac{1}{3}\arctan(u) + C_2 = \frac{1}{3}\arctan(x^3) + C_2$$

$u = x^3$

$\frac{du}{dx} = 3x^2$

$du = 3x^2\,dx$

$\frac{1}{3}du = x^2\,dx$

(b) $\displaystyle\int_0^{\pi/3} \tan^3 x \sec^2 x\,dx = \int_0^{\sqrt{3}} u^3\,du = \frac{1}{4}u^4\Big|_0^{\sqrt{3}}$

$$= \frac{1}{4}(\sqrt{3})^4 - \frac{1}{4}(0)^4$$

$$= 9/4$$

$u = \tan x$

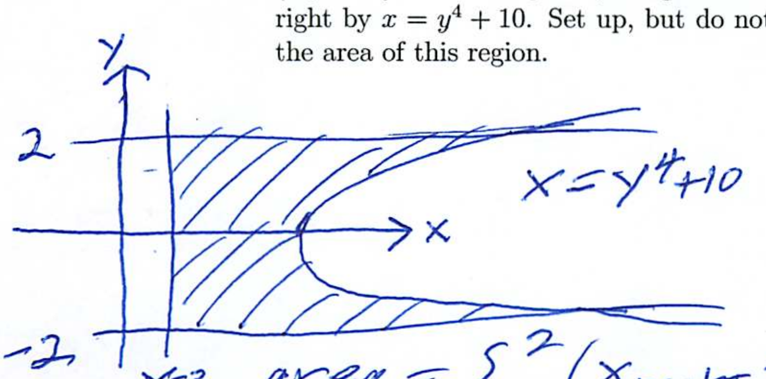
$\frac{du}{dx} = \sec^2 x$

$du = \sec^2 x\,dx$

$x = 0 \Rightarrow u = \tan(0) = \frac{0}{1} = 0$

$x = \frac{\pi}{3} \Rightarrow u = \tan(\pi/3) = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$

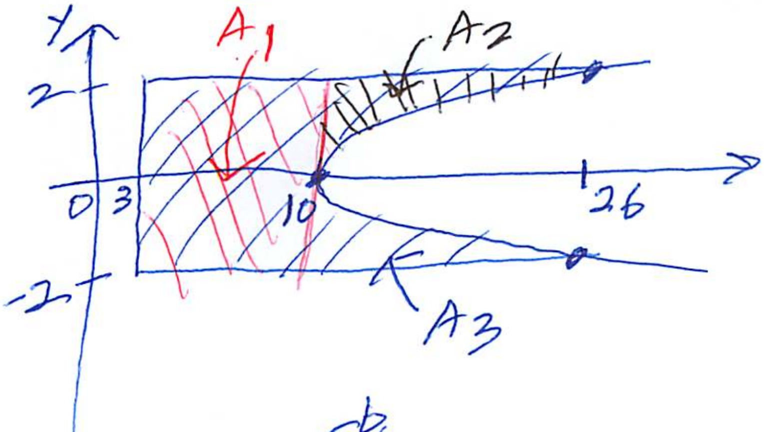
4. (2 points) For $-2 \le y \le 2$, a region is bounded on the left by $x = 3$ and bounded on the right by $x = y^4 + 10$. Set up, but do not evaluate, one or more integrals which represent the area of this region.



$$\text{area} = \int_{-2}^{2} (x_{right} - x_{left})\,dy$$

$$\text{area} = \int_{-2}^{2} ((y^4+10) - 3)\,dy = \int_{-2}^{2} (y^4+7)\,dy$$

④ another solution



$x = y^4 + 10$

$y = \pm \sqrt[4]{x-10}$

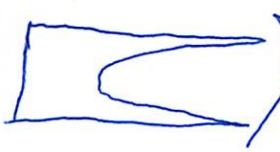
area $= \int_a^b (y_{top} - y_{bottom})\,dx$ for each region shown

area $= A_1 + A_2 + A_3$

$= \int_3^{10} (2-(-2))\,dx + \int_{10}^{26} (2 - (\sqrt[4]{x-10}))\,dx + \int_{10}^{26} ((-\sqrt[4]{x-10}) - (-2))\,dx$

OR
many other approaches due to symmetry as well as the fact that some regions are simply rectangles.

For example

area() = area() − area()

= area() − 2·area()

$= 23 \cdot 4 - 2\int_{10}^{26} \sqrt[4]{x-10}\,dx$