MATH 220
Test 1
Spring 2011

Name _____________________________
(Solutions)

(circle your TA discussion section)

• AD1, TR 1:00-1:50, Sarah Son
• AD4, TR 1:00-1:50, Sogol Jahanbekam
• AD7, TR 3:00-3:50, Nersés Aramyan
• AD9, MW 9:00-10:50, Ben Reiniger
• AD2, TR 1:00-1:50, Daniel Hockensmith
• AD5, TR 2:00-2:50, Daniel Hockensmith
• AD8, MW 11:00-12:50, Austin Rochford

• Sit in your assigned seat (shown below).
• Do not open this test booklet until I say START.
• Turn off all electronic devices and put away all items except a pen/pencil and an eraser.
• You must show sufficient work to justify each answer.
• While the test is in progress, we will not answer questions concerning the test material.
• Quit working and close this test booklet when I say STOP.
• Quickly turn in your test to me or a TA and show your Student ID.

263 264 265 266 267 268 269 270 • 271 272 273
240 241 242 243 244 245 246 • 247 248 249 250 251 252 253 254 255 • 256 257 258 259 260 261 262
217 218 219 220 221 222 223 • 224 225 226 227 228 229 230 231 232 • 233 234 235 236 237 238 239
194 195 196 197 198 199 200 • 201 202 203 204 205 206 207 208 209 • 210 211 212 213 214 215 216
171 172 173 174 175 176 177 • 178 179 180 181 182 183 184 185 186 • 187 188 189 190 191 192 193
148 149 150 151 152 153 154 • 155 156 157 158 159 160 161 162 163 • 164 165 166 167 168 169 170
• • • • • • • • 139 140 141 142 143 144 145 146 147 • • • • • • • • • 16 133 134 135 136 137 138
93 94 95 96 97 98 99 • 100 101 102 103 104 105 106 107 108 • 109 110 111 112 113 114 115
70 71 72 73 74 75 76 • 77 78 79 80 81 82 83 84 85 • 86 87 88 89 90 91 92
47 48 49 50 51 52 53 • 54 55 56 57 58 59 60 61 62 • 63 64 65 66 67 68 69
24 25 26 27 28 29 30 • 31 32 33 34 35 36 37 38 39 • 40 41 42 43 44 45 46
• • • • • • 17 18 19 20 21 22 23

FRONT OF ROOM – 314 Altgeld Hall
1. (10 points) Given the function \( f(x) = \frac{(\ln x)^3 - 6}{5} \), find a formula for its inverse function \( f^{-1}(x) \).

\[
\text{Let } y = f^{-1}(x) \\
f(y) = x \\
\frac{(\ln y)^3 - 6}{5} = x \\
(\ln y)^3 - 6 = 5x \\
(\ln y)^3 = 5x + 6 \\
\ln y = \sqrt[3]{5x + 6} \\
y = e^{\sqrt[3]{5x + 6}} \\
f^{-1}(x) = e^{\sqrt[3]{5x + 6}}
\]

2. (10 points) Find the domain of the function \( f(x) = \ln \left( 5 - \sqrt{12 - 4x} \right) \).

\[
12 - 4x \geq 0 \\
-4x \geq -12 \\
x \leq 3
\]

\[
5 - \sqrt{12 - 4x} > 0 \\
5 > \sqrt{12 - 4x} \\
25 > 12 - 4x \\
4x > -13 \\
x > -\frac{13}{4}
\]

Domain: \( \left( -\frac{13}{4}, 3 \right] \)
3. (10 points) Determine the value of $k$ for which $f(x)$ is continuous throughout its domain.

\[ f(x) = \begin{cases} \frac{\sin x}{x} & \text{for } x < 0 \\ 4e^x - k & \text{for } x \geq 0 \end{cases} \]

\[ \frac{\sin x}{x} \text{ is continuous for } x < 0 \]

\[ 4e^x - k \text{ is continuous for } x \geq 0 \]

we must check at $x = 0$.

\[ \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{\sin x}{x} = 1 \]

\[ \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (4e^x - k) = 4 - k \]

\[ f(0) = 4 - k \]

To be continuous at 0, we must have that \( 1 = 4 - k \)

so \( k = 3 \)

4. (5 points) Which one of the following statements is true?

(a) A function which is continuous at all points in its domain must be one-to-one.
(b) A function which is one-to-one must be increasing on its domain.
(c) A function which is continuous at a point $a$ must also be differentiable at $a$.
(d) A function which is differentiable at a point $a$ must also be continuous at $a$.
(e) A function which is differentiable at a point $a$ must be differentiable at all other points in the domain of the function.
(f) A function which is continuous at a point $a$ must be continuous at all other points in the domain of the function.
5. (12 points) Let \( f(x) = \frac{6}{x} \). Use the definition of a derivative as a limit to prove that \( f'(x) = -\frac{6}{x^2} \).

Show each step in your calculation and be sure to use proper terminology in each step of your proof.

\[
\begin{align*}
  f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
  &= \lim_{h \to 0} \frac{\frac{6}{x+h} - \frac{6}{x}}{h} \\
  &= \lim_{h \to 0} \frac{6x - 6(x+h)}{x(x+h)h} \\
  &= \lim_{h \to 0} \frac{-6}{x(x+h)} \\
  &= \frac{-6}{x(x+0)} \\
  &= \frac{-6}{x^2}
\end{align*}
\]
6. (10 points) Carefully sketch a graph of \( f(x) = 3 \cos (x - \pi/2) \) on the interval \([0, 2\pi]\). Be sure to label the tick marks along the \( x \)-axis and \( y \)-axis so that the coordinates of important points on your graph are clearly shown.

7. (6 points) What is the value of \( \sin \left(2 \tan^{-1}(3)\right)\)? Write your answer as either a simplified fraction or in decimal form.

\[
\sin(2 \tan^{-1}(3)) = 2 \sin \theta \cos \theta
= 2 \left(\frac{3}{\sqrt{10}}\right) \left(\frac{1}{\sqrt{10}}\right) = \frac{6}{10} = \frac{3}{5}
\]

8. (7 points) Determine real numbers \( a \) and \( b \) so that the expression \( \frac{6 - 4 \sin^2 \theta}{\cos^2 \theta} \) can be rewritten as \( a \tan^2 \theta + b \).

\[
\frac{6 - 4 \sin^2 \theta}{\cos^2 \theta} = \frac{6 - 4 \sin^2 \theta}{\cos^2 \theta}
= 6 \sec^2 \theta - 4 \tan^2 \theta
= 6 \left(\tan^2 \theta + 1\right) - 4 \tan^2 \theta
= 6 \tan^2 \theta + 6 - 4 \tan^2 \theta
= 2 \tan^2 \theta + 6
\]

So \( a = 2 \), \( b = 6 \).
9. (5 points each) Evaluate the following limits and simplify each answer. Show sufficient justification for each answer. An answer of ‘does not exist’ is not sufficient. If the limit is infinite then you must state if it is $\infty$ or $-\infty$.

(a) $\lim_{x \to 2} \frac{3x^2 - 2}{x^2 + 4} \rightarrow 10$

\[
\lim_{x \to 2} \frac{3x^2 - 2}{x^2 + 4} = \frac{10}{8} = \frac{5}{4}
\]

(b) $\lim_{x \to 3^+} (800 - 4\ln(x - 3))$

\[
\lim_{x \to 3^+} \ln(x - 3) = -\infty
\]

\[
\lim_{x \to 3^+} (800 - 4\ln(x - 3)) = +\infty
\]

(c) $\lim_{x \to 2/3} \frac{9x^2 - 4}{3x - 2} \rightarrow 0$

\[
\lim_{x \to \frac{2}{3}} \frac{9x^2 - 4}{3x - 2} = \lim_{x \to \frac{2}{3}} \frac{(3x - 2)(3x + 2)}{3x - 2}
\]

\[
= \lim_{x \to \frac{2}{3}} (3x + 2)
\]

\[
= 3\left(\frac{2}{3}\right) + 2 = 4
\]
(d) \( \lim_{x \to \pi/2} \frac{5 \sin^2 x}{4 \cos^2 x} \to 5 \)

\[
\lim_{x \to \pi} \frac{5 \sin^2 x}{4 \cos^2 x} = \infty
\]

(e) \( \lim_{x \to \infty} \frac{(2x + 1)^3}{6 - 5x^3} \)

\[
\lim_{x \to \infty} \frac{8x^3 + 12x^2 + 6x + 1}{6 - 5x^3}
\]

\[
= \lim_{x \to \infty} \frac{(8x^3 + 12x^2 + 6x + 1) \cdot \frac{1}{x^3}}{(6 - 5x^3) \cdot \frac{1}{x^3}}
\]

\[
= \lim_{x \to \infty} \frac{8 + \frac{12}{x} + \frac{6}{x^2} + \frac{1}{x^3}}{\frac{6}{x^3} - 5}
\]

\[\frac{8}{-5} = -\frac{8}{5} \]

(f) \( \lim_{x \to 0} \left( \frac{2}{x} - \frac{32}{x^2 + 16x} \right) \)

\[
\lim_{x \to 0} \left( \frac{2(x + 16)}{x(x + 16)} - \frac{32}{x(x + 16)} \right)
\]

\[
= \lim_{x \to 0} \left( \frac{2x + 32 - 32}{x(x + 16)} \right)
\]

\[
= \lim_{x \to 0} \frac{2x}{x(x + 16)}
\]

\[
= \lim_{x \to 0} \frac{2}{x + 16} = \frac{2}{16} = \frac{1}{8}
\]
Students – do not write on this page!

1 (10 points) ___________________

2 (10 points) ___________________

3 (10 points) ___________________

4 (5 points) ___________________

5 (12 points) ___________________

6 (10 points) ___________________

7 (6 points) ___________________

8 (7 points) ___________________

9a (5 points) ___________________

9b (5 points) ___________________

9c (5 points) ___________________

9d (5 points) ___________________

9e (5 points) ___________________

9f (5 points) ___________________

TOTAL (100 points) _____________